OHoC
373.1909713
059DE/C-M

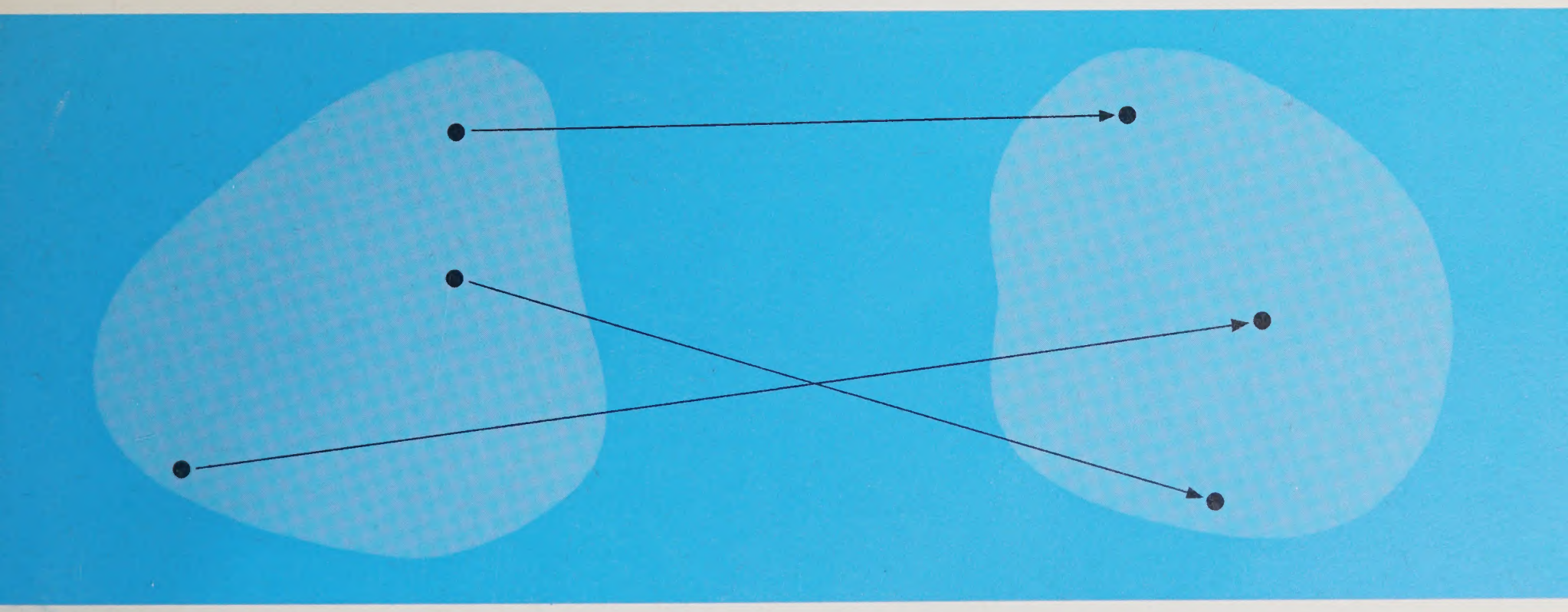
Mathematics

Ministry of Education
Ontario

Senior Division

Authorized by the Minister
of Education

1972



Digitized by the Internet Archive
in 2020 with funding from
University of Toronto

https://archive.org/details/seniordivisionma00mini

Mathematics

This curriculum guideline outlines the mathematics program for the Senior Division of the schools of Ontario. It provides a basis for the development of specific programs by local committees and individual teachers, and needs to be amplified with more specific resources from a wide variety of sources. The program consultants of the Ontario Department of Education can be of considerable assistance in the process of local course development.

This document provides a rationale for a diversified program in order that student needs and abilities may be taken into account when planning local courses and, also, that the broader needs of the community in both the local and global sense may be met by graduates.

This curriculum guideline should be used, then, as a planning and resource guide. New courses based on this rationale should be phased into schools as teachers, singly and in local committees, complete the necessary planning and preparation.

Senior Division Mathematics replaces the following documents:

S.12A and S.12A Adaptation

S.12B

S.12C and S.12C Supplement

R.P. 12 (pages 5-8, 10-13)

R.P. 27 (pages 250, 254-258)

R.P. 31 (pages 42-44, 72, 73).

Additional copies of this document and program consultant assistance may be obtained from the regional offices, Ontario Department of Education.

Region 1, Northwestern
303 News Chronicle Building
Water Street
Thunder Bay, Ontario

Region 2, Midnorthern
1349 Lasalle Boulevard
Sudbury, Ontario

Region 3, Northeastern
P.O. Box 3020
North Bay, Ontario

Region 4, Western
759 Hyde Park Road
London, Ontario

Region 5, Midwestern
279 Weber Street North
Waterloo, Ontario

Region 6, Niagara
15 Church Street, Suite 402
St. Catharines, Ontario

Region 7, West Central
40 Eglinton Avenue East
Toronto 315, Ontario

Region 8, East Central
Heron's Hill Building, Suite 3201
2025 Sheppard Avenue East
Willowdale, Ontario

Region 9, Eastern
1082 Princess Street
Kingston, Ontario

Region 10, Ottawa Valley
1825 Woodward Drive
Ottawa K2C OR2, Ontario

Contents

Rationale 3

Introduction
Mathematics and Society
Curriculum and Program
Recent Developments in School Mathematics

The Mathematics Program 5

Introduction
Applications of Mathematics 1, Applications of Mathematics 2
general information
outlines of topics
charts
Foundations of Mathematics 1, Foundations of Mathematics 2
general information
outlines of topics
sets, logic, proof, structure, problem-solving
transformations and vectors
charts
New or Experimental Courses
Honour Graduation Courses
general information
charts

Developing Courses 17

Introduction
A Diversified Program
Scope
Applications
Preparing for Innovations
Evaluation
Conclusion

Rationale

Introduction

Today's student is bombarded with information from the mass media. Whether aware of it or not, he is frequently making value judgements on this avalanche of information. It is increasingly apparent that the school must recognize the existence of these other educational forces and incorporate them in its planning. Unless this is done, a major gulf may develop between the world the student knows and the program that the school offers In addition, the burgeoning of a technological society and closer contact with distant parts of the world contribute to the student's educational needs. Every student is subject to many influences; those closest to him are best qualified to judge the resultant needs that he brings to the educational system.

A basic assumption of this document is that the curriculum should help the student to understand his world and, through this understanding, prepare him to respond to its challenges and to contribute in some way to its advancement.

Mathematics and Society

Mathematics has played a vital role in the development of civilization, not only in such closely allied areas as the physical sciences and the technologies based on them, but also in the development of such less obviously related fields as commerce, economics, geology, geography, medicine, biology, art, music, and the social sciences. Recent applications can be found throughout the entire spectrum of human endeavour, ranging from such unexpected areas as linguistics, coding patterns, simulations, strategies of war and peace, agriculture, and educational games and toys, to such obvious areas as computer science, computer technology, and space travel. Computers, with their capacity for high speed calculation and ability to store and retrieve vast amounts of information, have made it possible to explore mathematical problems whose solutions have hitherto been inaccessible to the human mind.

It is impossible to anticipate all the applications of the mathematics of tomorrow. Nevertheless, it is certain that mathematics will play an even greater role in man's quest to understand and control his environment. Consequently, more and more people will need to understand and apply mathematics in the pursuit of their careers.

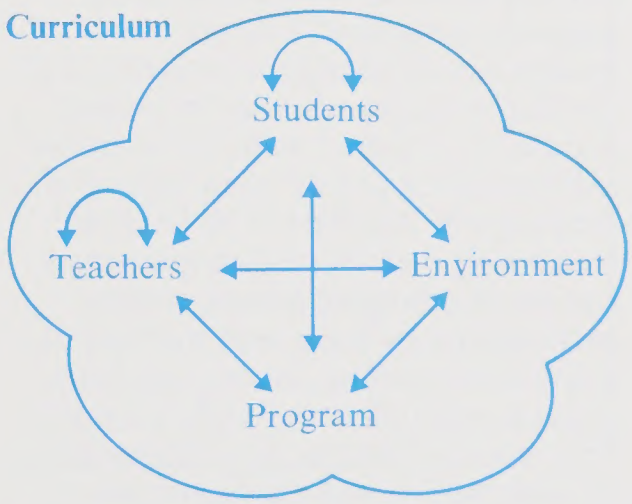
It follows that the study of mathematics should be an integral part of the education of many students in the Senior Division. Through his study of mathematics the student should develop an appreciation of its contributions to the development of civilization, of its potential, and of its patterns of thought. It is axiomatic, however, that students bring differing facility to the understanding of mathematical ideas and that their prospective careers present widely differing requirements. In order to appeal to students of diverse backgrounds and ambitions, the mathematical experiences available should be large in scope and varied in purpose.

Curriculum and Program

Since various connotations are often associated with the words *curriculum* and *program*, it is desirable at this stage to describe the sense in which they are used in this document.

Curriculum is used to represent the totality of students' experiences in interaction with fellow students, with teachers, with the ideas being investigated, and with all other influencing factors from their environment—both in and out of school. *Program* is used to describe various aspects of the plan for effecting the curriculum. The *mathematics program* of a school has its foundation in the spectrum of mathematics courses and related activities that the school offers to its students. A particular course encompasses the detailed planning that takes place both in selecting the course content from the Departmental "outline of topics" and in determining the scope of these topics as they are adapted to local conditions.

This view of curriculum and program is illustrated in the diagram below which indicates the lines of interaction.



The major responsibility for curriculum clearly belongs in the school. The prime function of a central organization or a local curriculum committee is to provide resources to blend with conditions within the school and the community in order to create an environment in which the curriculum 'happens'.

Recent Developments in School Mathematics

The mathematics program for the Senior Division should evolve from the students' earlier experiences. This can only happen if those responsible for planning the program are aware of current developments in school mathematics at all levels.

In recent years, mathematics programs in the schools of Ontario, as throughout the world, have undergone extensive reforms. Primitive foundations of topics previously studied only at post-secondary levels are now introduced in many elementary and secondary schools. These innovations involve methods and approaches appropriate to the maturity of the students. With younger children, in particular, new programs stress intuition and include activities that encourage a subconscious understanding of important mathematical principles. The accumulation of these early experiences should help these students in later studies to understand a more formal organization of mathematics and its applications.

The necessity of maintaining student interest and responsibility in the learning situation is becoming more generally recognized. Current trends involve a more conscious recognition of the diverse composition of the student population, together with provisions for individualizing at least some aspects of the program. Teachers at all levels are broadening their classroom techniques and are encouraging greater student participation by using such methods as group investigations, student seminars, projects, and individual study programs. They are emphasizing an understanding of mathematics, making greater use of varied media (such as films, television, film loops, and concrete models), expanding the resources available to students, and relating mathematics to the environment by drawing mathematics from it and by emphasizing problems related to real situations.

At present there is a movement in the Primary and Junior Divisions towards broad programs that survey mathematics in the areas of number and operations, measurement and relationship, and shapes and space. Basic notions of number, algebra, two- and three-dimensional geometry, statistics, and graphs pervade investigations in which students explore situations in their environment and in which subject disciplines are often integrated. The demonstration of the need for arithmetic operations precedes their formal study, and the necessary practice to develop arithmetic skills is often included in games requiring mental calculations, in investigations, and in the recognition of number patterns. Although significant mathematical principles are at the foundation of the activities, there should be little emphasis on encouraging the students to make generalizations prematurely or to abstract from their experiences before they are ready to do so.

Thus, because the learning environment stimulates their interests and because they are led to explore a variety of situations, many students in the Primary and Junior Divisions acquire a range of experiences upon which a more significant study of mathematics can be developed in the Intermediate and Senior Divisions. If this interest is to be maintained, and basic notions enlarged upon, the secondary school mathematics teacher must have a clear understanding of the nature of the elementary school program.

In the Intermediate Division a conscious transition to a more formal understanding of mathematics begins. On the basis of their earlier experiences and their current investigations, students are now encouraged to make generalizations and to test these with further numerical cases. Their mathematical vocabulary begins to take on precision and basic aspects of mathematical structure start to emerge. Students expand their knowledge of numbers and their properties, numeration systems, geometry, and algebra. They also investigate basic notions of probability and statistics.

In the later stages of the Intermediate Division the program becomes more diversified. Certain aspects of it are suited to students who are comfortable with abstractions, proof, and a more structured organization of mathematics. For these students, basic ideas of vectors, transformations, and groups are introduced with an extension of the students' knowledge of number, algebra, and geometry. Conjectures are made on patterns that seem to emerge from investigations and the need for proof follows. Intuition and applications to concrete situations are still evident throughout this development and should be carried forward into the Senior Division. In many cases, specific applications of mathematics to other subject areas are introduced in consultation with the teachers of these subjects. Other aspects of the program are suited to those who understand mathematics best when it is developed from and applied to practical situations For them, less emphasis is placed on the emerging structures and the notions of proof.

The trends established in the later stages of the Intermediate Division extend into the Senior Division, where many students have developed more specialized interests. Those who are planning further study and application of mathematics beyond secondary school should be encouraged to broaden the scope of their studies and to deepen their understanding in preparation for the exacting demands to be expected in post-secondary schools.

In summary, this document supports the premise that curriculum is constantly changing and that yesterday's mathematics is not sufficient for the society of today. A truly current program should anticipate the needs of tomorrow. It should build on the student's background and be continually updated. Thus the ultimate planning of the mathematics program should be done in the school itself. Only here can courses be adapted from year to year in designing programs that will appeal to students and meet local needs; only here can innovations in methodology and in mathematical approaches be tried with the opportunity for instant assessment and change, when change is desirable; only here can the program in mathematics be adapted to the facilities and resources available and to the programs in other subject areas.

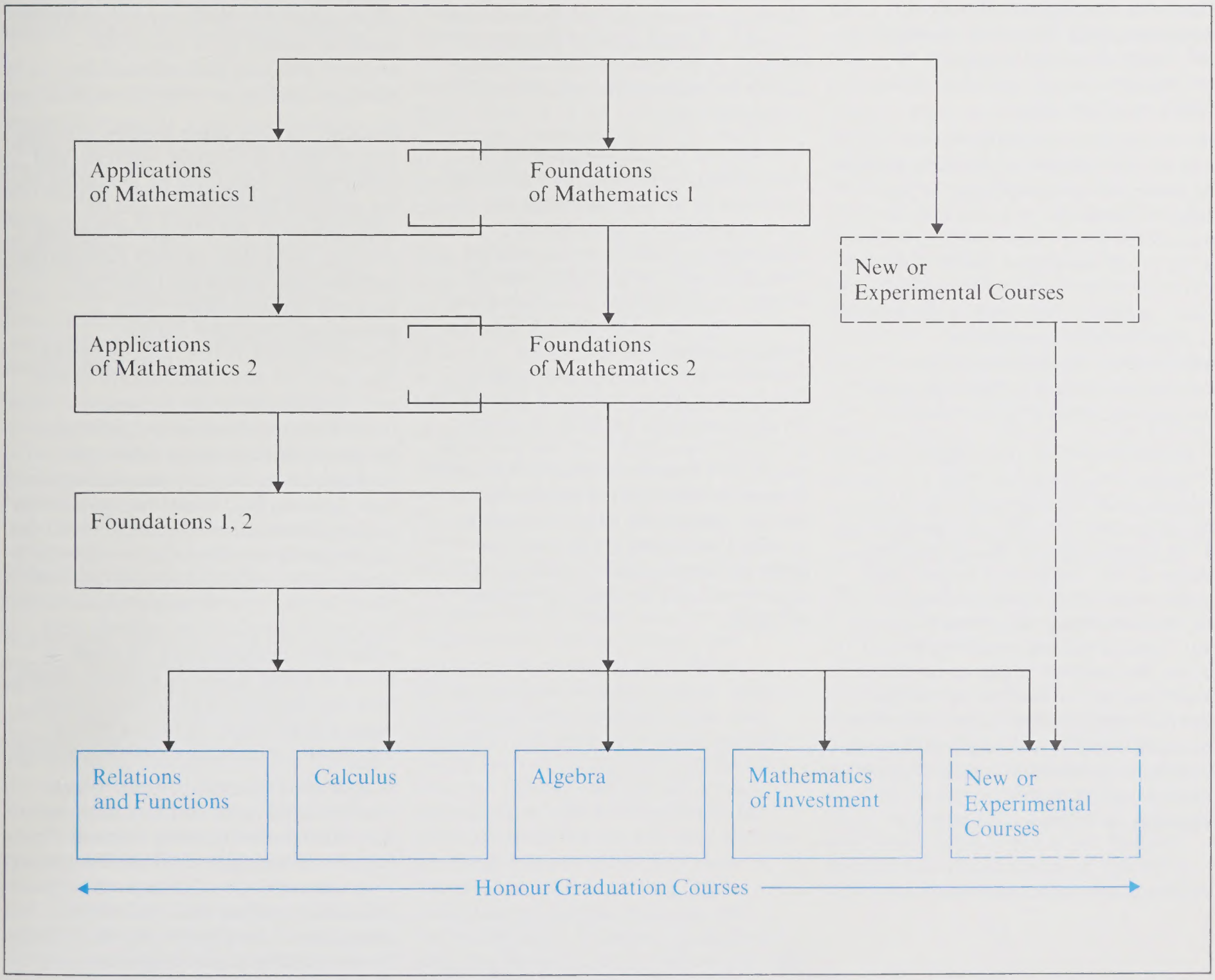
The Mathematics Program

Introduction

The school's mathematics program in the Senior Division should reflect the basic needs and interests of the students for whom it is intended as well as the concerns of the community at large. It is therefore expected that each school will offer a diversified program within the rationale of this document. The first steps toward this program are taken when the school develops a variety of courses based on the "outlines of topics" named in the chart below. To provide the schools with assistance in this phase of local planning, the general nature of the provincial outlines of topics is discussed in this section and factors to be considered during the local development of courses are discussed in the Section *Developing Courses*.

The program may be augmented, as local conditions suggest, by courses of a new or experimental nature. These courses may be planned by school staffs to meet the needs or interests of special groups of students. Some of these may pioneer general program changes of the future.

The Program for Senior Division Mathematics



Each of the outlines in *Applications of Mathematics* and *Foundations of Mathematics* is quite general and contains many more topics than would normally be studied in a single credit course. This provides great latitude for developing from each outline a number of courses that differ both in content and in scope.

The *Applications of Mathematics* outlines are intended as a guide for planning courses for students who understand mathematics better when it grows out of and is applied to practical situations. For these students, experiences with the nature of proof and the fundamental structures of mathematics are not vital for future studies or career plans.

The *Foundations of Mathematics* outlines are suitable for planning courses for those who are preparing for the study of mathematics at the honour graduation level. These courses, both in the topics developed and in their potential for providing the student with a growing awareness of the nature of proof and the structure of mathematics, should provide the foundations needed to meet the more exacting demands of future studies.

In planning courses at the Senior Division level, these two pairs of outlines should not be considered as mutually exclusive. This is suggested in the above chart by the overlapping nature of the "outlines of topics" and by the open-ended frames. Many of the characteristics associated with one pair, both in the suggested topics and in the emphasis to be placed on their development, can be used to enrich courses based upon the other pair. For example, while developing foundations for honour graduation courses, many students would benefit from an emphasis on related applications.

These applications may well include topics and approaches suggested by the *Applications of Mathematics* outlines.

On the other hand, practical applications of topics listed only in the *Foundations of Mathematics* outlines such as transformations or matrices, may be of interest to some students in courses based on the *Applications of Mathematics* outlines.

As a general rule, the nature of courses based on either the *Foundations of Mathematics* or the *Applications of Mathematics* outlines will be such that a student should proceed sequentially through them. Exceptions to this rule should be made by the school on an individual basis.

Foundations 1, 2, as indicated in the above chart, is intended as a make-up course to be developed locally for students with a background in *Applications of Mathematics* courses who need additional preparation for courses in mathematics at the honour graduation level. This course should comprise those topics from the *Foundations of Mathematics* outlines that these students need as a foundation for more advanced studies. It should provide a transition from the practical emphasis of earlier courses to the more theoretical emphasis of courses at the honour graduation level. Permission to offer this course is not required from the Department of Education, since it is based on existing outlines of topics.

Applications of Mathematics 1, Applications of Mathematics 2

General Information

Courses based on *Applications of Mathematics 1* and *Applications of Mathematics 2* should be designed for students who are primarily interested in practical applications of mathematics, particularly in the business or industrial world. The prime emphasis in these courses should be on intuitive understanding derived from concrete experiences. Although notions of proof could be of value to some students, a practical approach to the applications of mathematics in areas of special interest to the student is of paramount importance. His mathematics, wherever possible, should grow out of his environment and be reinforced by applications that are relevant to him.

Students electing these courses will have a wide range of abilities, interests, and future needs. Some will be preparing for the study of mathematics and its applications at the colleges of applied arts and technology in such fields as the technologies, technician training, business administration, data processing, computer science, accounting, marketing, and insurance. The needs of these students are varied and they should be encouraged individually to develop in depth the particular mathematical skills and understanding that will benefit them in their later studies. Some may go directly into the business or industrial world; the mathematical needs of these students are generally less demanding and they will likely benefit most from a program that surveys the applications of basic principles with opportunities for study in depth in areas of special interest. Others may be uncertain of their future plans; these students may be taking mathematics with no specific objectives in mind.

It should be remembered that many students with latent ability to abstract may be attracted to these courses because of their practical orientation. For some of these students it may be desirable to adjust their individual programs by varying the depth of study, by introducing topics from the *Foundations of Mathematics* outlines,

and by including aspects of proof and mathematical structure. This should be done, however, without losing the stress on practical applications which attracts students to these courses.

Students taking courses based on the *Applications of Mathematics* outlines may develop an interest in mathematical studies at the honour graduation level. The transition can be achieved through additional studies suggested by *Foundations 1, 2*, by transferring to a course based on one of the *Foundations of Mathematics* outlines, or, if the student is ready to do so, by proceeding directly to honour graduation courses.

Most students will need to use mathematics in other subjects of their personal program. Teachers of other subjects should be consulted during the planning of courses so that frequent and explicit cross-over to these other subjects may be included. Identification of these other areas of application can make the program more relevant, and both subject disciplines will benefit. Some such subjects are physics, electricity, machine shop, geography, and business finance.

Some students may lack proficiency in basic skills, probably due to a lack of practice based on an understanding of fundamental principles. Such students require patient assistance; if given opportunities to grasp these principles and to experience success on a moderate scale, they will often develop new interest and competence.

In planning *Applications of Mathematics* courses, the emphasis should be on developing a solid foundation of mathematical principles related to practical applications wherever possible. The mathematical goals should include the acquisition of a reasonable facility and understanding in basic algebra, graphing in the plane, functions, and statistics, all reinforced by topics suited to the student's special concerns and related to his future plans. Current practices de-emphasize pencil and paper computational skills in involved problems in favour of the use of business machines, computational devices, and tables of values. In this light, extensive drill on operations with large numbers is outdated in a modern program.

Outline of Topics

In order to facilitate the local design of a variety of flexible courses adapted to the needs of a wide range of students, the outlines of topics for *Applications of Mathematics* contain a small core of basic topics and a wide selection of optional sections. Each of these outlines is arranged in three units in which each unit offers a minimal program of basic mathematics and related applications. (See chart, page 8.) The core contains the fundamental mathematical principles needed in the study of many of the options. The options are indicated by a blue background. Those on a faint blue background involve applications that are closely related to the core or earlier options, whereas those on the darker blue background have less association with the core and consequently could be developed in a non-sequential order. In selecting options from the chart, it should be noted that an understanding of each section may depend on background work of sections to its left or above, but not to its right or below.

There are more options in each unit than can be discussed in a given term, so that options related to the special interests of the students will have to be selected. Essentially the options fall into three categories: those suited for students with technical interests, such as trigonometry, vectors, statics, and geometry; those suited to students taking business subjects, such as interest, promissory notes, present value, and annuities; and those of a more general nature such as commission, perimeter, area and volume, income tax, stocks, and selections and arrangements. In some instances the teacher may wish to introduce options other than those in the outlines, such as coding, logic, mathematical games, matrices, and transformations. Provision is made for adapting the program in this way by the "topic of the teacher's and/or student's choice" at the foot of each unit.

Not only are the topics in each unit related in the sense that many of the options exhibit applications of the same basic principles contained in the core, but many of the topics introduced in the first unit reappear in other units so that continuity of topics may be maintained throughout both courses. For example, sections related to trigonometry and vectors, analytic geometry, and interest are found in each unit, so that the program can be adapted to include a study of any of these topics in an introductory manner or in graduated depth.

Since there is no prescription as to how many options should be studied in any one term, no course based on these outlines need be too long. A 'rule of thumb' might be to plan less than a full term's work for each unit with additional topics in reserve if the timing proves realistic. In addition, most of the sections in both outlines of topics are optional, so that options omitted in earlier units may be discussed in preference to options introduced in later units. In this connection, however, students' future plans should be carefully considered so that they may study options related to their individual interests and needs. In many instances, different groups within the class could study different options simultaneously. The fact that students are studying topics of their own choice with resources suitable for individual or group study should stimulate interest.

The order of the sections in the outline is not binding and many reorganizations are possible to meet special circumstances. For example, a student of electronics would benefit from an early intensive study of trigonometry; hence his program might stress this work early in the year. He would then apply these skills in his work in other subjects. Many students taking technical subjects will benefit from some of the business options as a preparation for operating their own businesses after leaving school.

	Applications of Mathematics 1			Applications of Mathematics 2		
Section	**Unit 1**	**Unit 2**	**Unit 3**	**Unit 1**	**Unit 2**	**Unit 3**
1	Algebra	Algebra	Radicals, Exponents	Sequences, Series	Polynomials	Statistics and Probability
2	Graphing	Systems of Equations	Exponential Function	Ordinary Annuities	Quadratic Function	Ellipse, Hyperbola
3	Straight Line	Distance	Compound Interest	Exponents, Logarithms	Parabola	Vectors in 3-space
4	Trigonometry	Vectors	Solution of Triangles	Trigonometry	Trigonometric Functions	Trigonometric Identities
5	Commission	Profit, Loss	Circle	Equations of Equivalence	Home Ownership	Customs, Excise Tax
6	Interest	Promissory Notes	Present Value	Annuities	Foreign Exchange	Bonds, Debentures
7	Perimeter, Area, and Volume	Municipal Taxation	Buying, Selling	Stocks	Centre of Gravity	Income Tax, T1 General
8	Geometry	Perimeter, Area, and Volume	Income Tax, T1 Short	Moment of Force	Sinking Funds	Friction
9	T.T.S.C.	Geometry	Coplanar Forces	Selections, Arrangements	Radian Measure	T.T.S.C.
10		Linear Programming	Geometry	T.T.S.C.	Complex Numbers	
11		T.T.S.C.	T.T.S.C.		T.T.S.C.	

Sections with a blue background are optional.
T.T.S.C. means topic of teacher's and/or student's choice.

Applications of Mathematics 1

Outline of Topics

Unit 1

Section 1.1 — **Review of basic algebra**

Brief discussion of the natural, whole, integral, rational and real numbers

Operations of multiplication, division, addition and subtraction with rational numbers; use of the form $\frac{a}{b}$; the meaning of inverse operations and equivalent fractions; simplification of complex fractions

Decimal form of rational numbers; periods, decimals, approximations; use of slide rule and 10-key calculators

The product of a monomial and/or a binomial; finding a polynomial; factoring expressions with a factor common to each term; distributive property

Solution of first-degree equations in one variable

Section 1.2 — **Graphing of data obtained by experiments**

Collection of data from experiments; representation of data by (a) tables of values, (b) graphs; discussion of curve of best fit for linear and non-linear relations; interpolation and extrapolation

Review of the concept of (a) relation and function, (b) direct and partial variation (see section 1.3) using the graphs of the above topics

Discussion of rate of change of the dependent variable with respect to the independent variable for (a) linear functions, (b) non-linear functions

Section 1.3 — **The equation of a straight line**

Slope of a straight line

Equation of a straight line developed from the definition of slope; equivalent forms
$Ax + By + C = 0$ and $y = mx + b$

Review of roles of m and b in $y = mx + b$ including special cases

Review of direct variation ($y = mx$) and partial variation ($y = mx + b$) related to examples of section 1.2

Parallel and perpendicular lines; problems

AM1

Section 1.4 — **Primary trigonometric functions**

Definitions of $\sin\theta^\circ$, $\cos\theta^\circ$, $\tan\theta^\circ$ in terms of x, y, r, where θ is the degree measure of the angle and $0 \leq \theta \leq 360$; numerical examples; the relationship $r = \sqrt{x^2 + y^2}$

Investigation of the graphs of the functions defined by $y = \sin\theta$, $y = \cos\theta$, $y = \tan\theta$ for $0 \leq \theta \leq 360$ (leading to (a) discovery that the function values are independent of r, (b) readers prepared tables of values, and (c) commercially prepared tables for $0 \leq \theta \leq 90$

Interpretations of the definitions of the trigonometric functions in terms of the opposite side, adjacent side and hypotenuse of a right-angled triangle (including special cases for 30°, 45°, 60°)

Solution of right triangles, and applications to problems

Section 1.5 — **Commission**

Commission to salesmen; straight, graduated scale, salary plus; graphical illustrations; problems

Commission to agents; buying, selling

Section 1.6 — **Simple interest**

Principal, interest, and amount; calculations, graphical illustrations, problems

Calculation of rate, time, principal, amount

Simple examples to illustrate the meaning of nominal annual rate and effective annual rate

Instalment payments; calculation of effective annual rate using the formula
$r = \frac{2Nf}{P(n+1)}$

Graphical illustration of the inverse relation between any two of r, t, and p (n = period's and the other variable are constant

Section 1.7 — **Perimeter, area, and volume**

Investigation of (a) volume dealing with perimeter, area, and volume of regular and irregular plane figures and solids

Recognition that formulas derived for some of the regular plane figures and solids but that other methods must be used for perimeter, area, and volume of irregular shapes and solids

Calculations using formulas; problems

Section 1.8 — **Inductive, and informal deductive geometry**

Review of congruency, parallel lines, angle sum of a triangle and exterior angle of a triangle, externally opposite angles, use of these relations to prove simple deductions

Sum of angles of a polygon (a) convex and (b) regular; inductively, construction of some regular polygons

Investigation of the relation of angle sizes with lengths of sides for scalene triangles; illustration of indirect proof

Investigations of parallelograms and triangles on the same (or equal) base(s) and between the same parallels. Deductions and constructions based on these relations

Pythagorean Theorem; various models to illustrate; discussion of some proofs

Section 1.9 — **Topic of teacher's and/or student's choice**

AM1

Unit 2

Section 2.1 — **Review of basic algebra**

Rational numbers and irrational numbers, notations, approximations

Formula, solving, substitution

Product of two binomials; square of a binomial; division of a trinomial by a binomial

Order relation for rational numbers

Solution of equations and inequations of the first degree in one variable

Section 2.2 — **Linear equations and inequations**

Graphs of linear equations and inequations in two variables; applications

Graphical solution of a system of two linear equations in two variables

Algebraic solution of a system of two linear equations in two variables

Problems solved by the methods of the above topics

Section 2.3 — **Distance between two points**

Distance between any two points in the Cartesian Plane

Mid-point of a line segment

Tests for collinear points

Area of a triangle

Problems

Section 2.4 — **Vectors in two-dimensional space**

Displacements represented by directed line segments and ordered pairs, equal displacements

A vector with magnitude and direction represented by a directed line segment and by an ordered pair $[a, b]$; scalar, position vectors

Equal vectors

Addition of vectors by a scale drawing of directed line segments and by ordered pairs, intuitive discussion of closure, commutativity, associativity, zero vector (identity), and the negative of a vector (inverse); subtraction

Multiplication by a scalar

Converting the magnitude and angular direction of a vector to the ordered pair representation, and vice versa

Problems involving force, velocity, navigation, and displacement

Properties of geometric figures developed with vectors

Section 2.5 — **Profit and loss**

Relationship between selling price, cost price, and (i) profit (ii) loss

Mark-up on cost, margin on sales, converting markup to margin and vice versa, markdown

Analysis of Profit and Loss Statement

Section 2.6 — **Promissory notes**

Examples to illustrate the meaning of face value, interest, and value at maturity

Discounting of notes: discount period, discount, and net proceeds on (i) non-interest bearing notes, (ii) interest bearing notes

Section 2.7 — **Municipal taxation**

Municipal revenue and expenditure

Real property taxes: assessment and mill rate for residential and industrial properties

Local improvement taxes

Business taxes

Discounts and penalties on payment of taxes

Section 2.8 — **Perimeter, area, and volume**

Extension of the work in section 1.7 to continue some of the experimental situations and investigations (a) the ratio of perimeters, area, and volume of regular and irregular shapes and solids

Section 2.9 — **Inductive and informal deductive geometry**

Definitions and properties associated with a circle

Comparison of a central angle with an angle at the circumference subtended by the same arc, numerical problems

Investigation of: (i) properties related to the angles of a cyclic quadrilateral; (ii) the angle between the tangent at a point on a circle and (a) the radius, (b) a chord through the point, (c) tangents from an external point, numerical problems

Constructions such as: (a) circumscribed and inscribed circles of a triangle, regular n-gons in a circle, tangent at a point on a circle, and tangents from a point outside a circle

Section 2.10 — **Linear programming**

Graphs of linear inequations (see section 2.2), the region determined by the intersection of the graphs of (i) two inequations, (ii) more than two inequations

Simple problems to illustrate the application of linear programming to find the maximum (or minimum) value of an expression of the form $mx + by$ subject to the constraints of the problem

Section 2.11 — **Topic of teacher's and/or student's choice**

AM1

Unit 3

Section 3.1 — **Radicals and exponents**

Radicals: multiplication, conventions with entire and mixed radicals; addition, subtraction; discussion of a radical as an irrational number, non-periodic decimal and decimal approximations

Conversion of expressions with radical denominators such as $\frac{\sqrt{2}x}{3}$, $\frac{\sqrt{3}}{\sqrt{2}-2}$ and $\frac{\sqrt{28}}{\sqrt{21}}$ to equivalent fractions with rational denominators, their decimal approximations

Powers of the form a^n, a^0, and a^{-n} where $n \in N$; Power Laws for integral exponents

Discussion of a^r where $r \in Q$; Power Laws for rational exponents

Simplification of expressions such as $2^3 \cdot 2^5$, $5^8 \div 5^3$, 8^3, evaluation of powers such as 10^3, 10^{-15}, 10^{-3}, 10^{0}

Section 3.2 — **Exponential function**

Graphs of $y = 2^x$, $y = 3^x$, $y = 10^x$ where $x \in R$

Discussion of exponential growth

Discussion of exponential decay

Use of the graph of $y = 10^x$ where $0 \leq x \leq 1$ to express numbers in exponential form with base 10; simple calculations involving multiplication, division, and powers using the graph to approximate the exponents

Discussion of the exponential nature of the C and D scales of the slide rule as related to the graph of $y = 10^x$, where $0 \leq x \leq 1$

Use of tables of exponents for powers with base 10; calculations in exponential form; naming the exponent as a logarithm

Section 3.3 — **Compound interest**

Graphs of equations such as $A = (1.07)^n$, $A = (1.02)25^n$, $A = (1.01)^n$ where $n \in R$; discussion of rate of growth of a sum of money under compound interest

Compound interest in the business world, bank rates as step functions

Compound interest tables

Time diagrams to illustrate problems involving amounts; use of the formula $A = P(1 + r)^n$ to find the amount; discussion of the case where the interest is compounded m times per year

Simple interest in the fractional part of a time period

Finding r or n from $A = P(1 + r)^n$, including interpolation from a graph or tables

Section 3.4 — **Solution of oblique triangles**

Identities such as $\sin\theta = \sin(180 - \theta)$; use of tables to determine the function values of obtuse angles

Solution of oblique triangles using vectors

Use of trigonometric tables extended for reflex angles

Force, velocity, and displacement problems

Equilibrium problems

$\sin^2\theta + \cos^2\theta = 1$

Development of the Law of Sines and the Law of Cosines

Solution of oblique triangles, problems

Section 3.5 — **The equation of a circle**

Definition of a circle

Equation of a circle

Properties of a circle developed analytically from numerical cases

Section 3.6 — **Present value**

Meaning of present value

Use of time diagrams to illustrate present value problems, and the use of the formula $P = A(1 + i)^{-n}$ to find the present value P

Use of graphs as in section 3.3 to find the present value of an amount

Use of present value tables, problems

Problems involving present value and amount, dated values, and equivalent sets of payments illustrated by a time diagram

AM1

Section 3.7 — **Buying and selling**

List price, trade discount, and net price

A single discount equivalent to a series of trade discounts

Rate of trade discount when given the list price and net selling price

Cash discount

Sales tax: federal, provincial

Problems based on markup, margin, discounts, and sales tax

Section 3.8 — **Income tax, T1 Short**

The T4 slip

The T1 Short Form, persons who may use it

Discussion of deductions, dependents, Canada Pension Plan, shares, Old Age Security Tax, exemptions

Problems based on T1 Short Form

Preparation of a completed form from sample data

Section 3.9 — **Statics: coplanar forces**

Resultant and equilibrant of two or more forces acting at a point illustrated by a sketch, determined by a scale drawing and by calculations, problems

Condition for equilibrium of three or more coplanar forces acting at a point

Resolution of a force: component vectors of a force in two mutually perpendicular directions, problems

Section 3.10 — **Inductive and informal deductive geometry**

Review of ratio and proportion related to line segments

Similar figures, applications of similar geometric figures in drafting, architecture, industry, etc.

Inductive and deductive study of
(i) the ratio of areas of triangles with equal altitudes or equal bases
(ii) the ratios in which a line parallel to one side of a triangle cuts the other two sides
(iii) equiangular triangles

Constructions to divide a line segment (i) into n equal parts (ii) into the ratio $a : b$ where $a, b \in R$

Section 3.11 — **Topic of teacher's and/or student's choice**

Applications of Mathematics 2

Outline of Topics

Unit 1

Section 1.1 Sequences, series

Example of sequences and their graphs in $N \times R$ (including arithmetic and geometric sequences) + conjectures about general terms

Arithmetic sequence: graph, general term, applications; arithmetic series, sum of first n terms, applications

Geometric sequence: graph, general term, applications; geometric series, sum of first n terms, applications

Section 1.2 Ordinary annuities

Compound interest and present value

An ordinary annuity discussed as a sequence of periodic payments, illustrated by a time diagram (see *Applications of Mathematics I*, section 3.6)

Amounts and present value of an annuity; calculation of each of these by use of calculator and by the sum of a geometric series using compound interest tables

Calculation of the periodic payment given the amounts or present value of an annuity

Section 1.3 Radicals, exponents, logarithms

The set of real numbers as the union of the set of rationals and the set of irrationals; periodic and non-periodic decimals, examples of irrational numbers, approximations

Operations with radicals (see *Applications of Mathematics I*, section 3.1) extended to include conjugate radicals and their use in rationalizing the denominator

Radical equations

Powers and Power Laws for rational exponents (see *Applications of Mathematics I*, section 3.1) extended intuitively to real exponents, graph of $y = 10^x$ with $x \in R$, exponential computation using base 10

Discussion of meaning of logarithm, the equivalence of $y = 10^x$ and $\log_{10} y = x$; the graph of $x = 10^y$, i.e. $y = \log_{10} x$, discussion of inverse functions

Computations with logarithms: multiplication, division, powers, including roots

Section 1.4 Trigonometry, vectors, and the solution of triangles

Primary trigonometric functions: graphs, trigonometric function tables and their use with obtuse and reflex angles (see *Applications of Mathematics I*, section 3.4, 3.5)

Reciprocal trigonometric functions: graphs, tables of values

Solution of right triangles

Solution of oblique triangles using vectors and/or the Law of Sines and the Law of Cosines

Area of triangle by means of both (a) and the formula $\sqrt{s(s-a)(s-b)(s-c)}$

Section 1.5 Equations of equivalence

Dated values and equivalent sets of payments illustrated by a time diagram (see *Applications of Mathematics I*, section 3.6)

Comparison date problems to show that different comparison dates may be chosen without affecting the value of the variable

Solution of problems using time diagrams and equations of equivalence

Section 1.6 Annuities

Ordinary annuities, use of annuity tables to calculate amount, present value, or periodic payment

Meaning of an annuity due and comparison with an ordinary annuity; applications, practical problems (using annuity tables) to find amount, present value, or periodic payment

Meaning of annuity deferred, calculations of amount, present value or periodic payment found from the difference of two ordinary annuities

Section 1.7 Stocks

Types of stock: preferred, common, par value, no-par value

Market value, buying and selling, brokerage and transfer tax

Graphing: high, low, close, volume

Dividends: preferred, common, yield, price earnings ratio

Buying on margin, selling short

Section 1.8 Statics: moment of force

Rotational effect of a force acting on a body

Moment of a force: principle of moments, problems with light and heavy rods

Parallel forces

Machines and mechanical advantage

Section 1.9 Pascal's triangle, selections and arrangements

Pascal's Triangle: investigation of sequences of numbers derived from Pascal's Triangle

Simple problems involving arrangement and selection

Section 1.10 Topic of teacher's and/or student's choice

Unit 2

Section 2.1 Polynomials

Product of binomials, factoring of expressions of the form $ax^2 + bx + c$; $a, b, c \in I$, $a > 0$

product of two polynomials, division of a polynomial by a polynomial

Reduction of rational expressions: product of rational expressions; division with rational expressions

Addition and subtraction with rational expressions

Solution of a quadratic equation by factoring, completing the square, and formula; discussion of the existence of non-real (imaginary) roots

Problems

Section 2.2 Quadratic function

Investigation of problems leading to tables of values, graph which are parabolic in shape (quadratic function), maximum or minimum value, symmetry

Graphical investigation of functions defined by equations of the form $y = ax^2$, $y = ax^2 + c$, and $y = ax^2 + bx + c$; discussion of: x-intercepts (zeros of the function) including the cases of imaginary intercepts and equal intercepts, vertex, axis, symmetry, vertex, maximum or minimum value

Problems involving maximum (or minimum) value of a quadratic function by means of a graphical solution and, possibly, an algebraic solution

Investigation of graphs of polynomial functions of the third and fourth degree, zeros of the function and the factor theorem, factoring of sum or difference of two cubes

Section 2.3 The parabola

Cones, cone sections, discussion of their occurrences in the environment

Construction techniques for a parabola

Equation of the parabola in the form $y^2 = 4px$, $x^2 = 4py$; "five-point" sketch

Applications

Solution of a linear-quadratic system of equations by graphical and algebraic methods

Section 2.4 Amplitude, periodicity, and phase shift of the trigonometric functions

Discussion of $x = r\cos\theta$, $y = r\sin\theta$, and $r = x\cos\theta + y\sin\theta$, where $\theta \in R$, rotational angles, periodic nature of the functions

Discussion of amplitude, period, and phase of the primary functions and of functions with form such as $y = a\sin(\theta + \alpha)$, $y = a\cos(b\theta)$, $y = a\sin(b\theta) + c$; recognition that $\sin\theta = \cos(\theta - 90)$, etc.; applications

Section 2.5 Home ownership

Initial costs: down payment, legal fees, mortgage fee, adjustments to purchase price, miscellaneous

Maintenance costs: taxes, insurance, heating, utilities, repairs, depreciation and appreciation, interest on purchase price, miscellaneous

Calculation of fair rental value

Mortgages: types; methods of repayment such as, (a) regular payment on principal with additional payment to cover interest (b) uniform monthly payment to cover interest and principal

Section 2.6 Foreign exchange

Need for foreign exchange in travel and trade, types of foreign currencies

Conversion of Canadian currency to foreign currency

Conversion of foreign currency to Canadian currency

Multiple conversions

Methods of payment: travellers cheques, credit cards, letters of credit, drafts

Section 2.7 Statics: centre of gravity

Centre of gravity illustrated with models; definition, calculation of centre of gravity by principle of moments, problems

Stability of a variety of solids; factors affecting the stability, applications, calculation of the critical angle for tipping a rectangular block or cylinder

Section 2.8 Sinking funds

Meaning, method of finding (a) amount, (b) present value, and (c) periodic payment

Sinking fund schedules

Accumulative value of a sinking fund at a given time (i) from the schedule and (ii) by calculation

Section 2.9 Radian measure, trigonometric functions, periodicity

Radian measure, $\alpha = r\theta$, where α, r, θ are the measures of arc length, radius, and angle in radians; $r = ra, \theta = a\alpha$ and related problems

Trig function values when the radian measure is such as $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, or $\frac{\pi}{2}$

Graphs of $y = \sin\theta$, $i = \cos\theta$, and $y = \tan\theta$, where $\theta \in R$, the distinction between $y = r\sin\theta$ and $y = \sin\theta$

Graphs involving a combination of two or more functions, for example $y = r\sin\theta + c\cos\theta$ by the addition of ordinates, periodic nature of such graphs

Graphical and algebraic solutions of trigonometric equations and systems of equations

Section 2.10 Complex numbers

Non-real (imaginary) roots of a quadratic equation, their significance with respect to the graph of $y = ax^2 + bx + c$

Set of Complex Numbers: as an extention of R, i factor, graphing in the Complex (Argand) Plane

Complex number represented by a position vector

Operations with complex numbers: definitions, illustrations by graphing position vectors

Trigonometric and polar forms of a complex number, conversions

Applications

Section 2.11 Topic of teacher's and/or student's choice

Unit 3

Section 3.1 Statistics and probability

Collection and organization of data: specific cases in which the entire class collects data, tabulates it, and uses intervals, frequency distributions, histograms, and frequency polygons to analyse it

Mean, median, mode

Discussion of the role of statistics in our daily lives

An introduction to the use of statistics in industrial quality control by testing all items and by sampling

Probability: meaning, range of possible values, empirical probability discussed using statistical data and from experiments using coins, dice, tacks, etc.

Law of large numbers and its implications

Standard deviation, particularly as applied to confidence limits and to quality control charts used to control product uniformity in industry

Probability by reasoning: flipping coins, throwing dice, other simple games of chance

Simple examples to illustrate the probability of (i) compound events, by listing possibilities (ii) either of two independent events (or) both of two independent events, use of tree diagrams

Mendelian genetics

Section 3.2 The ellipse and hyperbola

Conic sections: parabola (see section 2.3); ellipse, hyperbola

Definitions of an ellipse and a hyperbola

Construction techniques, cursive stitching; introduction of basic vocabulary; discussion of the limiting cases of an ellipse and hyperbola as the distance between the foci changes and as the sum or difference of the focal radii changes

Equation of an ellipse with centre the origin and foci on (r, 0) and (a, 0) yields discussed and briefly, use of major axis $2a$, minor axis $2b$, $b^2 + f^2 = a^2$; to show that the sum of the focal radii $= 2a$, and $a^2 = b^2 + c^2$

Problems related to the equation of an ellipse

Applications of the ellipse

Equation of the hyperbola with centre the origin and foci on (r, 0) x-axis (a) y-axis use of transverse axis $2a$, conjugate axis $2b$, $b^2 + f^2 = 2c$ to show that the difference of the focal radii $= 2a$ the use of $c^2 = a^2 + b^2$ to locate the foci geometrically, brief discussion of the asymptotes of a hyperbola and their role in sketching a hyperbola

Rectangular hyperbola, $x^2 - y^2 = \pm a^2$, $xy = k$

Problems related to the equation of a hyperbola

Applications of the hyperbola

Solution of linear-quadratic system of equations by graphical and algebraic methods

Section 3.3 Vectors in three-dimensional space

Co-ordinates of a point in 3-dimensional space, physical models of 3-space

A vector in 3-space represented by a directed line segment and by an ordered triple, models of vectors in 3-space

Equal vectors

Addition of vectors, the magnitude and direction of a vector (that is represented by an ordered triple), discussion of closure, commutativity, and associativity under addition, the zero vector, and the negative of a vector, subtraction

Multiplication by a scalar

Simple problems

Section 3.4 Trigonometry

Basic trigonometric identities: reciprocal, pythagorean, $\tan\theta = \sin\theta/\cos\theta$, $\cot\theta = \cos\theta/\sin\theta$; exercises

Expansion of $\cos(\theta \pm \alpha)$, $\sin(\theta \pm \alpha)$, $\tan(\theta \pm \alpha)$, $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, exercises

Section 3.5 Customs duties and excise taxes

Customs duties and excise taxes; tariff schedules

Calculation of ad valorem duty, specific duty, duty-paid value, excise tax, sales tax, dumping duty

Section 3.6 Bonds and debentures

Types

Methods of issue, par value, premium, discount

Buying and selling: market price, cost of bonds (a) excluding interest and (b) including accrued interest, and commission

Approximate yield

Present value of a bond on a given date prior to maturity

Section 3.7 Income tax — T1 General

T4 T5 T5 slips

T1 General Form, and its use

Review of deductions, other than applications of *Mathematics I*, section 3.8)

Discussion of profit and loss statement, capital cost allowance, rental income, investment income, capital gains

Problems based on T1 General Form

Preparation of a completed form from sample data

Section 3.8 Statics: friction

Friction: occurrences, advantages and disadvantages, factors that affect its magnitude

Normal force, coefficient of friction

Problems dealing with an object on a horizontal plane and on an inclined plane under the action of an external force parallel to the plane

Section 3.9 Topic of teacher's and/or student's choice

Foundations of Mathematics 1, Foundations of Mathematics 2

General Information

Courses based on *Foundations of Mathematics 1* and *Foundations of Mathematics 2* are intended for students who may study mathematics at the honour graduation level and possibly beyond, or who have an intrinsic interest in mathematics and some facility with abstractions. The student's background for these courses has been established throughout his earlier studies and is now expanded in the integrated program suggested by these outlines of topics. These courses, although flexible in many respects, are intended to blend a study of synthetic and analytic geometry, algebra, and various relations and functions with optional developments of topics such as matrices, probability and statistics, transformations, and vectors. This fusion of topics is illustrated below.

The shaded region in the diagram indicates the basic notions of number, sets, logic, proof, vector space, and other mathematical structures that are present throughout the development of these courses. It is significant that students absorb at least intuitive notions of these central ideas. Such experiences will prepare them for a more formal study of mathematics and its structures in honour graduation courses and at post-secondary institutions. This move towards developing the structures of mathematics, however, should not be emphasized at the expense of intuition or of applications of mathematics, nor should it be imposed on non-receptive minds. Not all students taking these courses need the emphasis on rigour, structure, and proof required by those who are planning an intensive study of mathematics at the university level.

It should be emphasized once more that courses based on the outlines for *Foundations of Mathematics* may also place a strong emphasis on practical applications. In this way, students with both a practical orientation and an interest in future mathematical studies may elect courses better suited to their personal needs.

Many provisions are suggested for a variety of interpretations of *Foundations of Mathematics 1* and *Foundations of Mathematics 2* to suit the wide range of abilities, interests, and needs of students. These involve not only the selection of topics, but the depth and breadth of study of these topics and the approaches and methodologies employed. They are discussed throughout this document to show ways in which these courses can be adapted for students who are studying mathematics for its own sake or who may need mathematics for future career plans involving:

- university studies in such fields as mathematics, mathematics and physics, engineering, physical sciences, biological sciences, architecture, social sciences, languages:
- studies at the colleges of applied arts and technology in such fields as technologies, data processing and computer science, technician training, business administration, accounting, marketing, insurance:
- careers in teaching, in accountancy, or in other fields requiring a mathematical background.

Units	Foundations of Mathematics 1 Sections	Foundations of Mathematics 2 Sections
Geometry (synthetic)	1.1 Congruence, Parallelism 1.2 Similar Figures 1.3 Area, Volume 1.4 Circle, Sphere	1.1 Loci 1.2 Circle, Sphere 1.3 Similar Figures 1.4 Geometries, Euclidean and Otherwise
Algebraic Operations	2.1 Polynomials, Rational Expressions 2.2 Real Numbers, Absolute Value, Radicals	(informal review as required)
Relations and Functions	3.1 Relations 3.2 Graphs	2.1 Relations and Functions 2.2 Graphs 2.3 Combinations of Functions
Quadratic Functions and Equations		3.1 Quadratic Functions 3.2 Quadratic Equations 3.3 Theory of Quadratic Equations 3.4 Rate of Change 3.5 Polynomial Functions
Analytic Geometry	4.1 Straight Line 4.2 Systems of Linear Equations 4.3 Properties of Geometric Figures 4.4 Circle	4.1 Loci 4.2 Straight Line 4.3 Circle 4.4 Parabola, Ellipse, Hyperbola
Trigonometric Functions	5.1 Trigonometric Functions 5.2 Applications	5.1 Trigonometric Functions 5.2 Applications 5.3 Polar Coordinates
Exponential and Logarithmic Functions		6.1 Exponential Function 6.2 Logarithmic Function 6.3 Business Applications 6.4 Slide Rule
Sequences and Series		7.1 Sequences 7.2 Series 7.3 Mathematical Induction
Probability and Statistics		8.1 Statistics 8.2 Probability

All topics in the light blue sections are optional
All topics in the dark blue sections are supplementary

Some topics in the other sections are optional or supplementary
(See charts between pages 14 and 15)

Outline of Topics

The outlines of topics for *Foundations of Mathematics* are organized into units and these into sections, with titles as indicated in the chart opposite. Some of the sections are marked as optional or supplementary; for the other sections, individual topics are designated as core, optional, or supplementary. On this basis, a number of courses can be developed from each outline. Each of these will have a common core of topics required for future studies but differing from each other in the optional and supplementary topics and in the scope of the development of all topics.

The *core topics* are printed on a white background. They constitute the basic ideas with which all students should be familiar by the time any course based on the outline is completed. As many of the core topics review and extend ideas found in previous courses, some students will need a less detailed development than others. These topics are repeated here to maintain continuity with the students' earlier experiences and, where necessary, to consolidate the background knowledge needed for other topics in the unit. Generally, it is assumed that those who understand the mathematics in earlier courses will spend about two-thirds of their time on the core program with the remaining time on in-depth study or on optional or supplementary topics. Others in the class may need more time on the core program or on fundamentals in preparation for it.

The *optional topics* are printed on a light blue background. They suggest experiences that will enrich the students' understanding of core topics through extensions of the core, alternative approaches to the core topics, or, in some instances, a preview of future topics. Their omission will not preclude the study of core topics in later courses but may detract from the student's total appreciation of mathematics and handicap his study of mathematics in later years.

The *supplementary topics* are printed on a darker blue background. They, too, lead to an enriched understanding of the core topics and are suitable for individual or group projects. Unlike the optional topics, they are not directly related to topics that occur in later mathematics courses in the Senior Division. Some schools have prepared lists of hundreds of supplementary topics from which students may make selections for essays, seminars, group discussion, individual study, or other projects.

When a course is organized so that a student spends part of his time on individual or group projects, his work should be evaluated on its own merit and anecdotal records maintained. These records can then be combined with the evaluation of the student's other work to form a more complete picture of his understanding of mathematics.

The description of the topics in each outline is deliberately non-prescriptive. In most instances, it gives no indication of a detailed development of the topics, of the breadth or depth of investigation, of the approach or approaches to be taken, or of suitable methodology. Decisions related to the scope of the program are expected to be made locally, where they may be more closely allied to the needs and interests of the students. The core topics are not necessarily intended to be developed by traditional approaches and methods, although this possibility certainly exists.

In many instances, the approach suggested by an optional topic makes a separate discussion of the core topics by traditional methods unnecessary. This is particularly true with respect to the study of either transformations or vectors and their applications. For example, if the study of similarity (in Section 1.2 of *Foundations of Mathematics 1*) is introduced through notions of dilatations and isometries, a traditional development is unnecessary.

The order in which the units, and the topics within them, are given does not necessarily indicate the order in which the topics should be developed. Many other orders are possible, including ones in which topics from different units are interwoven. For example, in *Foundations of Mathematics 1* (see chart **FM1** opposite page 14.) there are many advantages of combining certain aspects of:

- *Analytic Geometry* (Unit 4) with *Geometry* (Unit 1)

or

- Graphs of relations and functions (in Unit 3) with *Algebraic Operations* (Unit 2), *Analytic Geometry* (Unit 4), and *Trigonometric Functions* (Unit 5)

or

- Problems involving variation (in Unit 1, Section 3) with the linear function (in Unit 4, Section 1).

New sequences of topics can be readily developed by rearranging the topics in the outline. For example, one possible way of integrating topics from the five units of *Foundations of Mathematics 1* is given on the next page:

Integrated Trimester Outline

Term 1	Term 2	Term 3
1.1 Congruence and Parallelism (core, isometries, possibly vectors)	**4.2** Systems of Linear Equations (applications in **4.3**)	**4.3** Properties of Geometric Figures
2.2 Real Numbers (brief review of rationals, irrationals, denseness, continuity)	**2.1** Operations with Rational Numbers (continued)	**1.2** Area, Volume (core, shears, stretches, formulas)
3.1 Relations, Graphs **3.2** (core, (i), (ii))	**2.2** Real Numbers $y = \lvert f(x) \rvert$ **3.2** $y = \frac{1}{f(x)}$	**3.2** Circular Functions **5.1** Trigonometric Functions
4.1 Straight Line (core, translations, reflections, 90° rotations)	**1.3** Similar Figures (core, dilatations—synthetic and analytic, applications in **4.3**) variation	**5.2** Applications (core, possibly some options)
2.1 Operations with Rational Numbers (part of this section)		**1.1**, **1.3**, **4.1**, **4.3**, **5.2** Vectors and Applications (if time permits)

According to individual circumstances, the above plan could be further adapted to groups within the class by remedial work, review, and algebraic practice. The study of optional and supplementary topics other than those in the above plan could be included.

In the study of a course trimestered in the above manner, the students would have two or three exposures to ideas from each of the five units during the year, and possibly more if the exercises include opportunities for a further relating of topics. If ideas developed early in the year are kept in use, they become relevant to the students through application. This process may help the students to see mathematics as a complete fabric, rather than as the separate strands from which it is woven.

Sets, Logic, Proof, Structure, Problem-solving

The outlines for *Foundations of Mathematics 1* and *Foundations of Mathematics 2* contain few, if any, direct references to sets, logic, proof, structure, and problem-solving. This should not be interpreted to mean that these aspects of mathematics have no place in this program. On the contrary, they are so basic to the understanding of all the topics that it is assumed that notions of them will be present throughout the development of the program.

Sets

It is anticipated that the language, notations, and ideas associated with sets will be used in a natural way when they add precision to the discussion. The student should be familiar with the basic notions of sets, subsets, union, intersection, Venn diagrams, and the use of such symbols as:

$\{\ \}, \in, \notin, \subset, \supset, \cap, \cup$, and $\emptyset$.

At this level, it is generally advisable not to create artificial situations that escalate the study of sets for their own sake.

Logic

An understanding of certain basic notions of logic is necessary for accurate communication of ideas, both in mathematics and in everyday situations. Again, no formal study of logic is specifically suggested, but it is anticipated that fundamental aspects of logic will be introduced incidentally throughout this program in order to add clarity to the discussion. During his studies, the student should become familiar with the use of the modifier *not*, the connectives *and, or, if . . . then . . , only if, if, and only if* (*iff*), and with the meaning of such terms as *negation, complement, conditional, implication, converse, bi-conditional.* He should also acquire an intuitive understanding of universal and existential quantification.

Opportunities to apply these aspects of logic are present in all areas of this program and should not be confined to the study of geometry.

A knowledge of basic notions of sets and of logic provides the student with a foundation upon which a study of the algebra of sets and of logic may be developed and related to each other. Although this more formal study is probably best reserved for later courses, an unsophisticated awareness of these relationships could be established for some students at this time.

Proof

Throughout his studies in the Senior Division, it is assumed that the serious student of mathematics will have many opportunities to develop a deepening understanding of proof. He should learn various strategies for proof, and the significance of axioms through a local organization of axiomatic systems, and should be prepared to argue logically in defense of his mathematical assertions.

The student's experiences should help him to realize that a valid argument based on a false assumption yields a consequent that is not necessarily true. To illustrate, by assuming for any two real numbers a and b that $a > b$ implies $a^2 > b^2$, he can prove with $a = 2$ and $b = -5$. that $4 > 25$. The conclusion is false even though the logical argument is valid. On the other hand, with $a = 2$ and $b = 1$ by the same argument he can show that $4 > 1$, which is true.

Similar illustrations can be given in all phases of mathematics to show that the consequent of a valid argument is not necessarily true. Therefore, when a consequent of a valid argument is known to be false, we know the assumption must be false.

Structure

It is likely that the student will need many practical experiences before he can begin to appreciate the structure that lies beneath a broad study of mathematics. This appreciation can grow from situations in which repeated patterns suggest a common structure. It is for this reason that the unifying structures of mathematics should not be imposed or emphasized too early in the mathematics program. The student should also analyse situations in which a property does not hold in order that he may see value in finding those cases in which it does. For example, the property of commutativity is more significant when the student examines operations defined on sets for which it does not hold, and the property that the diagonals of a parallelogram bisect each other is more significant when he examines the diagonals of a quadrilateral in general.

The degree of emphasis placed on the above notions should be determined by local conditions. Students who are planning to study mathematics beyond the secondary school level, however, will benefit from an increased emphasis on these aspects.

Problem-solving

Problem-solving should be a central activity in any mathematics program. Although not specifically stated for each of the major topics in the outline, emphasis on solving problems and interpreting situations by mathematical representations (building mathematical models) is desirable throughout the program. The accumulation of a bank of interesting, realistic problems gleaned from a variety of sources would be a worthwhile project for both the staff and students of a school.

Transformations and Vectors

In recent years, revisions at all levels of pre-college curricula throughout the world have included notions of transformations and vectors. In Ontario, many elementary school programs now include activities related to slides, turns, flips, symmetry, and tesselations, in preparation for a more formal development of the isometries in later years. These early activities involve the use of cardboard cut-outs, tracings, mirrors, and compasses to study ideas related to physical motion. Investigations with geoboards and elastics or string, shadows, pinhole cameras, plasticine, and other pliable materials provide concrete experiences at this level, upon which a later study of shears, stretches, dilatations, projections, and other mappings of points can be built. The important notion of invariance is present in mathematics at all levels: the cardinality of a set of physical objects is invariant when the arrangement of the objects is changed: the numerical value of an arithmetic expression is invariant under certain operations; the roots of an equation are invariant under certain modifications in the equation (those that produce an equivalent equation), and certain points in the plane are invariant under reflection or rotation but not under translation.

Activities involving displacements and experiences with "strings of numbers" are also being used to introduce informal notions of vectors. Further, the programs of the Intermediate Division provide opportunities for extending these earlier experiences in order to discover many mathematical properties by an intuitive use of transformations and vectors.

In the Senior Division, a transformation approach to most of the topics would provide a unifying theme for the mathematics program. The geometric properties of two- and three-dimensional figures can be derived through applications of a synthetic study of isometries, shears, stretches, and dilatations. A corresponding development of point mappings (such as translations, reflections in the axes and in the line $y = x$, stretches, dilatations, and special rotations) provides a common basis for studying the graphs of functions and relations, and for developing the analytic geometry of the line, parabola, circle, ellipse, and hyperbola as well as the linear, quadratic, cubic, trigonometric, exponential, logarithmic, and slope functions. In addition, vectors, matrices, complex numbers, groups, and isomorphisms are closely related to transformations. This approach is readily adaptable to three-dimensional situations; in many instances, proofs of theorems developed for two-dimensional cases are quite similar to those that apply to their three-dimensional analogues. Cogent reasons exist, therefore, for incorporating these important ideas in the program.

Thus, by using a transformation approach, the student can become aware of relationships linking many topics from separate branches of mathematics and can develop a consistent approach that leads to simple and often elegant proofs. Further, transformations are suitable for all levels of classroom development, from intuitive investigations with concrete materials to a formalistic-axiomatic study. Consequently, this approach is readily adaptable to a wide variety of classroom situations.

Recent projects related to linear algebra indicate that a similar development of the mathematics program is possible along these lines. Several years of introductory experiences with vectors in two- and three-dimensional spaces would culminate in the study of the real inner-product vector space. However, unless the foundation of a linear vector space is fully understood, the topic could easily become too abstract.

Although there are many valid reasons for including the study and application of transformations and vectors in *Foundations of Mathematics 1* and *Foundations of Mathematics 2*, their use is optional at this time for the following reasons:

- the non-prescriptive nature of the outlines is intended to give a choice in determining the approach taken for development of the topics;
- transformations and vectors, being relatively new in the secondary mathematics program, require experimentation in content and methodology;
- textual material appropriate for dealing with these topics in secondary schools is limited at present;
- many teachers will need time for personal study and assessment before they are ready to fully commit themselves to using either approach.

Foundations of Mathematics 1

Outline of To...

Unit 1 Geometry

Section 1.1 Congruence, parallelism

Problems involving congruence, parallelism, angle relationships, inequalities, relations of a triangle, indirect proof

Extensions from two- to three-dimensional situations wherever feasible

Applications of vectors and transformations to develop the above topics

Properties of vectors under 'addition' and the isometries under composition, groups, matrices

Solids and their nets, Euler's relation, the five regular convex polyhedra

Section 1.2 Similar figures

The meaning of ratio, division of a line segment in a given ratio, equality of two ratios, the meaning of proportion

Equations equivalent to $\frac{a}{b} = \frac{c}{d}$

Problems from science and technology involving variation

The concept of dilatation in two-space and in three-space, its properties, applications, matrices

Proof of properties of dilatation

The meaning of similar figures

Area of a triangle is proportional to its base and to its altitude, triangle parallel-projection theorem, sufficient conditions for similar triangles, similarity in three-dimensional situations

Applications of vectors and transformations to develop the above topics, matrices

Section 1.3 Area, volume

Parallelograms, triangles, prisms and pyramids on equal bases and with equal altitudes

The concepts of shear, stretch, and dilatation related to area and volume, matrices

Pythagorean Theorem and its application to two- and three-dimensional figures

Formulas for the mensuration of rectilinear figures, the circle, cylinders, cones, and the sphere, applications

Section 1.4 Circle, sphere

Definitions and symmetry properties associated with the circle and the sphere

Chord properties of a circle

Relation between a central angle and an angle at the circumference of a circle subtended by the same or equal arcs, angles in the same segment of a circle, in a semi-circle, and in a cyclic quadrilateral

Applications of vectors and transformations to develop the above topics

Geometry of the sphere: great circles and other plane sections, latitude, longitude, distances on the earth, projections

Unit 2 Algebraic Operations

Section 2.1 Polynomials, rational expressions

Recall of addition, subtraction, multiplication, and division with rational numbers in fractional form

Multiplication and division with polynomials

Factorization of algebraic expressions, the factor theorem, applications to (a) solution of equations, and (b) operations with rational expressions

Factorization of the sum or differences of cubes, factoring by completing the square

Section 2.2 Real numbers, absolute value, radicals

Recall of the decimal form of rational and irrational numbers, the set of real numbers, order and completeness in the real numbers

Cardinality and completeness in the set of real numbers

Absolute value

Solution of equations and inequations involving one variable and absolute value

Equations and inequations in which two of the terms involve absolute value

Radicals: operations with second order radicals, including examples with variables under the radical sign

Discussion of radicals of order greater than two

Radical equations

Unit 3 Relations and Functions

Section 3.1 Relations

$R \times R$, subsets of $R \times R$ and their graphs

Meaning of relation, function, domain, range, and intercepts

Functions with domain (i) N, (ii) R, density and continuity

Section 3.2 Investigation of the graphs of some relations

Given the graph of $y = f(x)$, techniques for sketching the graph of

$y = cf(x)$
$y = f(x) + d$ where c and d are constants
$y = cf(x) + d$
$y = \frac{1}{f(x)}$

Applications of point mappings to the topics of this section

Inverse of a relation

Properties of relations determined from their graphs

Suggested relations for the above investigations

(i) Linear relations
(ii) Quadratic relations
(iii) Circular functions
(iv) Polynomial functions of degree greater than two
(v) Exponential functions
(vi) Functions continuous almost everywhere
(vii) Relations defined by equations such as
$y = \sqrt{x - [x]}$, $y = \sqrt{25 - x^2}$, $|y| = x$,
$x^2 + y^2 = 16$, $2x^2 + y^2 = 16$, $x^2 - y^2 = 16$
(viii) Step functions

Unit 4 Analytic Geometry

Section 4.1 The straight line

The linear function

Equation developed from the fundamental property

Equation of the form $Ax + By + C = 0$, including cases where one or more of A, B, C, is zero

Condition for parallel lines and for perpendicular lines, applications

Proofs of the conditions for parallel and perpendicular lines

Applications of transformations to the topics of this section

Family of lines: (i) of a given slope, (ii) parallel to the y-axis, (iii) through a point, parameters

Section 4.2 Systems of linear equations

Solution of a system of two linear equations in two variables by (i) a graphical method, (ii) algebraic methods

Consistent and inconsistent systems

Systems of equations with literal coefficients

Solution of a system using matrices

Cartesian 3-space: points, planes, lines, graphs

Solution of a system of three linear equations in three variables

Linear programming

Section 4.3 Properties of geometric figures

Mid-point of a line segment

Point of division of a line segment in a given ratio

Position vector of the point of division

Study by analytic geometry of properties related to parallelism, perpendicularity, lengths of segments, special concurrent lines, and points of division

Area of a triangle

Applications of vectors and transformations to the topics of this section

Development of the formula for distance between a point and a line

Problems involving the distance between a point and a line

Section 4.4 The circle

Equation of a circle with centre the origin

Domain, range, and symmetry

Problems involving chords

Unit 5 Trigonometric Functions

Section 5.1 The trigonometric functions

Investigations leading to the graphs and definitions of the sine, cosine, and tangent functions

Properties of the functions derived from their graphs

Applications of the graphing techniques of Unit 3, section 2 using the trigonometric functions

The inverses of the trigonometric functions, graphs and properties

Images of the graphs of the functions under transformations

Applications of the sinusoidal curve

Section 5.2 Applications

Degree and radian measures of angles, conversions

Use of tables of the trigonometric functions

Interpretation of the function definitions for right triangles, solution of right triangles, problems

The reciprocal trigonometric functions definitions, applications to the solution of triangles

Simple identities

Development of the sine and cosine laws, problems involving oblique triangles

A sector described as an ordered pair and in terms of its magnitude and direction, conversions from one form to the other, solution of oblique triangles using vectors in ordered pair form, proofs of the sine and cosine laws using vectors

Dot product of two vectors, applications

The graphs and properties of the reciprocal functions

Solution of problems involving more than one right-triangle, two- and three-dimensional configurations

Projections, geometric theorems related to the cosine law, Pythagorean Theorem as a special case of the cosine law

Unit 5	Trigonometric Functions
Section 5.1	**The trigonometric functions**
	Investigations leading to the graphs and definitions of the sine, cosine, and tangent functions
	Properties of the functions derived from their graphs
	Applications of the graphing techniques of Unit 3, section 2 using the trigonometric functions
	The inverses of the trigonometric functions, graphs and properties
	Images of the graphs of the functions under transformations
	Applications of the sinusoidal curve
Section 5.2	**Applications**
	Degree and radian measures of angles, conversions
	Use of tables of the trigonometric functions
	Interpretation of the function definitions for right triangles, solution of right triangles, problems
	The reciprocal trigonometric functions: definitions, applications to the solution of triangles
	Simple identities
	Development of the sine and cosine laws, problems involving oblique triangles
	A vector described as an ordered pair and in terms of its magnitude and direction, conversions from one form to the other; solution of oblique triangles using vectors in ordered pair form, proofs of the sine and cosine laws using vectors
	Dot product of two vectors, applications
	The graphs and properties of the reciprocal functions
	Solution of problems involving more than one right-triangle, two- and three-dimensional configurations
	Projections, geometric theorems related to the cosine law, Pythagorean Theorem as a special case of the cosine law

Foundations of Mathematics 2

Outline of Topics

Unit 1 Geometry

Section 1.1 Loci

Investigation of the set of points determined by a given condition or conditions in 1-, 2- and 3-space

Section 1.2 Circle, sphere

Definition and elementary properties associated with a circle and a sphere

Perimeter and area of a regular n-gon if $n = \infty$ circumference and area of a circle, area of a sector of a circle

Chord, angle, secant, and tangent properties associated with circles

Area of a segment of a circle, surface area and volume of a sphere

Chord, angle, secant, and tangent properties associated with spheres

Applications of vectors, transformations, and their properties to the topics of this section

Locus problems

Circumcircle, inscribed and escribed circles of a triangle, orthocentric, centroid

Nine point circle, Euler Line, Simson Line, other famous problems of geometry

Circular inversion

Paper folding, curve stitching, and model building to demonstrate properties related to circles and spheres

Applications in Geography: great circles, latitude, longitude, distances on the earth, projections

Section 1.3 Similar figures

Conditions for similar polygons and polyhedrons

Areas of similar polygons, surface areas and volumes of similar polyhedrons, applications

Mean proportional theorem, its application to proving Pythagoras' Theorem

Applications of vectors and transformations to develop the topics of this section

Mechanical linkages to draw transformation images

Section 1.4 Geometries: Euclidean and other types

Unit 2 Relations and Functions

Section 2.1 Relations

The concepts of relation and function, common descriptions and notations for these concepts

Classes of relations: 1-1, many-1, 1-many, many-many, onto, into

Inverse of a relation

Composition of functions

Section 2.2 Investigation of the graphs of some relations

Given the graph of $y = f(x)$, techniques for sketching the graph of

$y = cf(x)$
$y = f(x) + d$
$y = f(kx)$
$y = f(x - h)$
$y = f^{-1}(x)$
$y = cf(k(x + h)) + d$
$y = \frac{1}{f(x)}$

where h, c, d, k are constants

Applications of point mappings to determine image of the graph of $y = f(x)$

Suggested relations for the above investigations

(i) Linear relations
(ii) Quadratic relations
(iii) Circular functions
(iv) Polynomial functions of degree greater than two
(v) Exponential functions
(vi) Functions continuous almost everywhere
(vii) Sequences
(viii) Relations defined by equations such as

$y = \pm\sqrt{3x - 2}$, $y = \pm\sqrt{25 - x^2}$,
$y = \pm\sqrt[3]{9 - x^2}$, $y = \pm\sqrt{x^3 - 1}$,
$x = |y - 1|$, $x = (y^2 - 2)$

Section 2.3 Combinations of functions

Where g and h are functions, the graph of f where f is

$h \circ g$
$g + h$
$g \cdot h$
$\frac{g}{h}$

Unit 3 Quadratic Functions and Equations

Section 3.1 Quadratic functions

Investigation of the quadratic function and its properties

Inverse of the quadratic function

Problems involving maximum or minimum values

Graphs of: i) $y = 1/f(x)$, ii) $y = \frac{1}{f(x)}$

where f is a quadratic function

Regions defined by quadratic relations

Applications of point mappings to the above topics

Section 3.2 Quadratic equations

Zeros of the quadratic function, roots of corresponding quadratic equations obtained by factoring, completing the square, and formula, problems

Solution of linear-quadratic systems, problems

Solution of quadratic inequations

Discussion of non-real roots, complex numbers

Quadratic equations with a variable such as x^2, $x - 2$, 2^x, and

Equations such as $\sqrt{4 - 3x} - x = 12$,

$\frac{4}{x + 1} + \frac{5}{x + 2} = \frac{12}{x + 3}$

Section 3.3 Theory of quadratic equations

Nature of the roots related to the discriminant

Sum and product of the roots, applications

Section 3.4 Rate of change

Intuitive discussion of rate of change, tangent at a point

Applications

Problems involving maximum or minimum values

Section 3.5 Polynomial functions of degree n, where $n > 2$

Investigation of polynomial functions and their properties, approximations of the zeros

Factorization, remainder theorem and factor theorem, complex zeros

Unit 4 Analytic Geometry

Section 4.1 Loci

Investigation of the set of points determined by a given condition or conditions in 1-, 2- and 3-space

Section 4.2 The straight line

Equations of a straight line, condition for parallel lines and perpendicular lines, family of lines through a point, for a given slope, and parallel to the y-axis, parameters

Solution of a system of (i) two linear equations in two variables, (ii) three linear equations in three variables

Distance between a point and a line

Cartesian Space: points, planes, lines, graphs

Solution of a system using matrices

Area of a triangle

Applications of vectors and transformations to the above topics

Section 4.3 The circle

Circle with centre the origin, symmetry, domain, range, intercept

Chord properties

Intersection of a straight line and a circle

Tangent(s) to a circle with centre the origin

Circle with centre (h, k) and radius r, problems

Sphere with radius r and centre (i) $(0, 0, 0)$, (ii) (h, k, m)

Applications of vectors and transformations to the above topics

Section 4.4 The parabola, ellipse, hyperbola

The parabola, ellipse, and hyperbola in standard positions with vertex or centre (i) $(0, 0)$, (ii) (h, k), properties, problems

Solution of linear-quadratic systems

Tangents to the curves above

The use of transformations to develop the above topics

Unit 5 Trigonometric Functions

Section 5.1 Trigonometric functions

Definitions of the trigonometric functions, graphs and properties of the primary trigonometric functions

Graphs and properties of the reciprocal trigonometric functions

Application of particular cases of transformations of the form $(x, y) \to (ax + b, cy + d)$ to trigonometric functions

Amplitude, periodicity, and phase shift related to the graphs of equations such as $y = c \sin(kx + h)$

Inverses of the trigonometric functions, their graphs and properties

Graph of a function formed by the combination of two functions

Identities

Trigonometric equations solved by graphs and/or tables of values

Applications of the sinusoidal curve

Section 5.2 Applications

Development of law of sines, law of cosines, problems involving the solution of triangles

Solution of problems involving more than one triangle, including three-dimensional configurations

Solution of oblique triangles using vectors

Dot product of two vectors, applications

Section 5.3 Polar coordinates

Polar coordinate system, graphs of equations of the form $r = f(\theta)$

Complex numbers in polar form

Unit 6 Exponential and Logarithmic Functions

Section 6.1 Exponential function

Graphs of equations such as $y = 2^x$, $y = 3^x$, and $y = 10^x$

Definition of a^x where x is (i) natural, (ii) integral, (iii) rational $a > 0$, to establish consistent Laws of Exponents

Rational approximants of a^x when x is real and $x > 0$, graphs of $y = a^x$ when $x \in R$ and (i) $a > 1$, (ii) $a = 1$, (iii) $0 < a < 1$, applications to exponential growth and exponential decay

Manipulative practice involving powers with rational exponents, equations involving exponents

Scientific notation

Computations using exponential tables for $y = 10^x$, when $0 \leq x \leq 1$

Section 6.2 Logarithmic function

The graph of the inverse of an exponential function, definition of logarithm

Conversion between equations of the form $x = a^y$ and $y = \log_a x$

Laws of logarithms

Computations using logarithms

Conversion formula: $\log_b a = \frac{\log_c a}{\log_c b}$

Section 6.3 Business applications

Simple interest, $A = P(1 + ni)$, graphs

Compound interest, $A = P(1 + i)^n$, graphs

Equivalent annual rate

Present value, $P = A(1 + i)^{-n}$

Section 6.4 The slide rule

Construction

Unit 7 Sequences, Series, Mathematical Induction

Section 7.1 Sequences

Definitions, notations, general term, recursion formulas, graphs

Limit

Arithmetic sequences, applications

Geometric sequences, applications

Fibonacci sequence

Section 7.2 Series

Definitions

Sigma notation

Inductive investigation of the sum of the first n terms of a series

Formulas for the sum of the first n terms of (i) arithmetic series, (ii) geometric series

Convergent and divergent series, limit of S_n as $n \to \infty$

Formula for the sum of a convergent geometric series

Annuities, amount and present value, mortgages, bonds, debentures

Continued fractions

Section 7.3 Mathematical induction

Proofs by mathematical induction

Unit 8 Statistics and Probability

Section 8.1 Statistics

The use and abuse of statistics

Collection, organization, representation, and interpretation of data

Mean, median, mode, and standard deviation

Section 8.2 Probability

Empirical probability: meaning, range of values, investigation of experimental and real life situations

Theoretical probability: simple and compound events

New or Experimental Courses

Some mathematics-oriented students may wish to extend their study of mathematics beyond the scope of the regular courses based on the provincial outlines of topics. There are a number of ways of meeting the needs of these students.

One possibility involves expanded courses based on additional topics from the *Applications* or *Foundations* outlines, or both—possibly studied in greater depth. Such courses should not be considered experimental and if they involve additional hours of study they can be pro-rated for credit value greater than one.

Another possibility exists for students who have strong interests and/or needs in more specialized branches of mathematics than those described in the existing outlines. For these students, the school may plan new or experimental courses which may be taken independently of, or in conjunction with, the courses in the regular program. Some experimentation of this type is now being conducted in the province in full courses related to probability and statistics, Euclidean geometry using the methods of Euclid in conjunction with transformation and vector approaches, and the applications of mathematics. Shorter courses of less than one credit value are also possible in areas such as the history of mathematics or statics. Other areas of possible experimentation involve mathematical model-building, queueing problems, networks, logic, Boolean algebra, mathematics seminars, mathematical games and puzzles, mathematics and the desk-top calculator or desk-top computer, mathematics and the computer, and computer-assisted instruction.

Care should be taken to ensure that students who are planning to take the regular honour graduation courses acquire the necessary foundations. Whereas a student may be able to proceed to a new or experimental course at the honour graduation level based on previous experimental courses, the regular courses at this level presuppose a background knowledge of topics in the *Foundations of Mathematics* outlines.

Approval to offer new or experimental courses for credit purposes must be obtained from the Department of Education through the procedure described in *Circular HS1*.

Honour Graduation Courses

General Information

Each of the honour graduation courses deals with a more specialized content than the earlier courses. Thus at the upper level of the Senior Division the student is able to select courses that are more suited to his personal needs. *Relations and Functions*, *Calculus*, and *Algebra* are composed of units from *Curriculum S.12C* (1966) reorganized into single-credit courses as in *Curriculum S.12C Supplement* (1971). *Mathematics of Investment* is taken from *Curriculum RP-31* (1963). The contents of these courses, although reorganized, have not been revised at this time; thus local adaptation of their scope is desirable so that they may be consistent with the revisions in the earlier courses of the program.

Sections with a blue background are optional.

R&F

Relations and Functions

Honour Graduation Level

Outline of Topics

This outline of topics contains [illegible] mathematics taken without reworking from *Curriculum 5/12A*. It is intended for use in the local planning of a course based on the mathematics in the core units and augmented by some of the topics in the optional units.

This course should take into consideration the students' earlier experiences with the topics in this outline. For example, many of the core topics can be developed from transformations and matrices if the students' background includes these topics. The course may be adapted to students interested in the social sciences or the business world by the choice of options on probability and statistics or mathematics of investment respectively. The level of scholarship and the depth of treatment should maintain the standards traditionally associated with grade 13 mathematics.

Some of the optional units in this outline are also listed in *Calculus of Algebra* because certain of their aspects are related to the core of these other courses. Repetition should be avoided in deciding where and with what emphasis these ideas are to be developed.

Unit 1 Function as a Mapping

Section 1.1 Functions

Function defined as a mapping of the domain onto the range; in particular, real valued functions of a real variable

The notation $f: x \to f(x)$ that is, the function f that maps x onto $f(x)$

Equivalence of this definition to the alternative definition in terms of ordered pairs

One-to-one mappings

Section 1.2 Inverse of a function

The inverse of a function

The condition that the inverse of a function is also a function

The relationship between the graph of a function and the graph of its inverse

Practice in determining the domain and range of the inverse of a function

Unit 2 Second Degree Relations in the Plane

Section 2.1 Graphs of second degree relations

A study of the graphs of relations defined by equations such as $x^2 + y^2 = r^2$, $x = ay^2$, $4x^2 + 9y^2 = 36$, $4x^2 - 9y^2 = 36$, $xy = 36$, $4y^2 - 9x^2 = 0$

Graphing and naming of the corresponding curves and their relationship to the sections of a cone

Determination in each case of domain, range, intercepts, and symmetry

Study of the corresponding inequalities and their graphs

Section 2.2 Equations of conics

Development of the equations of the parabola, ellipse and hyperbola from the focus-directrix definitions

Development of the equations of the ellipse and hyperbola from the constant sum and constant difference definitions

Study of the equations of the conics in the other standard positions, $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$, $x^2 - y^2 = a^2$, $xy = k$, $y^2 = 4px$, and $x^2 = 4py$ (for positive and negative values of p)

Examples of the occurrence in applied mathematics of the parabola, the ellipse, and the hyperbola

Section 2.3 Intersections

Intersections of lines with conics

Equations of tangents to particular conics determined by using the conditions for equal roots

Regions defined by the conjunction or disjunction of simple linear-quadratic inequalities

Intersections of conics with conics

Unit 3 Trigonometry

Section 3.1 The trigonometric functions

A review of the sine and cosine regarded as real valued functions of a real variable

The definition of the related functions tangent, cotangent, secant and cosecant and a study of the domain, range, and graph of each function

Section 3.2 Expansion identities

Development of the formulas for the sine, cosine and tangent of the sum and of the difference of two or real numbers

The trigonometric functions of $\pi + \theta$, $\frac{\pi}{2} + \theta$, $-\theta$

Development of the formulas for $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$ in terms of functions of θ

Some practice with trigonometric identities

Section 3.3 Phase shift, period, amplitude

A study of the phase shift, period and amplitude associated with $y = a\sin(kb + d)$ and $y = a\cos(kb + d)$

Study of the sum $a\sin\theta + b\cos\theta$

Development of the expression for the angle between any two lines in terms of their slopes

Unit 4 Transformations in the Plane

Section 4.1 Translation

Translation developed as a mapping of the point (x, y) onto the point $(x + h, y + k)$; use of notation $(x, y) \to (x + h, y + k)$

Graphical determination of images of points, lines, and simple polygons under translation

Development that distance is invariant under translation

Observation that the measure of an angle is invariant under translation

Deduction from these facts that the image of the locus of any equation has the same geometric character as the given locus

Practice in determining the images under translation of the relations defined by

$x^2 + y^2 = r^2$, $y^2 = 4px$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Determination of the translation that will eliminate the first degree terms in equations of the form $x^2 + y^2 + 2gx + 2fy + c = 0$

Section 4.2 Rotation

Rotation developed as a mapping of the point (x, y) onto $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ where θ is the measure of the rotation about the origin

Graphical determination of the images under rotation of points, lines, and simple polygons where the angles of rotation are multiples of $\frac{\pi}{4}$ or $\frac{\pi}{6}$

Verification for particular cases that distance and angles are preserved under rotation

Section 4.3 Reflection

Reflection in the x or y axes developed as a mapping of the point (x, y) onto $(x, -y)$ or $(-x, y)$ respectively

Graphical determination of images of points, lines, and polygons under reflection

Observation that distances and angles are invariant under reflection

Section 4.4 Study of $ax^2 + 2hxy + by^2 + gx + 2fy + c = 0$

Determination of the translation that will eliminate the first degree terms in equations of the form $ax^2 + by^2 + 2gx + 2fy + c = 0$

Intuitive determination of the form of the graph in various cases according as $ab = 0$, $ab > 0$, $ab < 0$

Determination of the rotation that will eliminate the xy term in equations of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Section 4.5 Two-way stretch, dilatation

Study of particular cases of the transformation $(x, y) \to (ax, by)$

Observation that when $a = b$ angles will be preserved, ratio of lengths of corresponding line segments will be preserved, but distances will not be preserved

Study of particular cases where $a \neq b$ to illustrate the distortion resulting from such transformations, e.g., the image of $x^2 + y^2 = 1$ under $(x, y) \to (ax, by)$

Unit 5 Matrices and Linear Transformations

This unit can be used to relate matrices to the transformations studied in Unit 4: both of these concepts can then be applied to a variety of situations throughout the core study. The topics on groups can be introduced by transformations, synthetically and analytically, and by matrices. Topics marked with an asterisk can be adapted to such a development.

Section 5.1 Matrices

- Definition of an m by n matrix $A = (a_{ij})$; m, n arbitrary but not too large), row and column vectors, definition of AX where X is n by 1
- Linear equations of the form $AX = B$
- AX as a function of X, statement and proof of linear properties

Section 5.2 Matrices and transformations

- Examples of rotation, reflection, projection, shear, identity, zero
- Product AB defined as composition, the rule $AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)$, observation that AB generalizes AX and that dot product is a matrix multiplication

Section 5.3 Properties

- Associativity, non-commutativity, identity, and zero divisors
- Definition of A^{-1} and its calculation, interpretation of A^{-1} as the matrix of the inverse of the linear transformation with matrix A, non-invertible matrices

Section 5.4 Groups

- Definition of group
- Investigation of various transformation groups
- Use of two-dimensional models of regular polygons to establish symmetry group characteristics

Examination of the systems of natural numbers, integers, rational numbers, real numbers and complex numbers to determine which systems form groups

- Examination of sets of functions that form groups ($f \circ g$ defined as $f(g)$ and use of f^{-1} as the group inverse)

Permutation developed as a one-to-one mapping of a set onto itself

Non-commutative groups

- Representation of various transformations by 2×2 matrices, investigation of group properties, transformation groups and the corresponding matrix groups

Identification of the characteristic properties of rings and fields, examples of rings and fields

- Transformations of a regular tetrahedron and cube

Unit 6 Complex Numbers and Polar Coordinates

Portions of this unit can be related to Units 2 and 3: complex numbers with the intersections of curves, the graphs of some equations in polar form with the conics, and de Moivre's Theorem with trigonometry and polar coordinates. Topics marked with an asterisk can be adapted to such a development.

Section 6.1 Complex numbers

- Complex numbers as ordered pairs and in the form $a + bi$

Field properties of the complex numbers

- Solutions of quadratic equations

Geometric form of complex numbers, absolute value, complex conjugates

- Polar form of complex numbers

Section 6.2 Polar coordinates

- Establishment of the correspondence for the point P between the rectangular coordinates (x, y), the polar coordinates (r, θ), and the rectangular components of the vector OP
- Practice at conversions from any one of these pairs to the other two
- Practice in graphing of relations expressed in polar form, particularly $r = a$, $r = a\cos\theta$, $r = a\sin\theta$, $r = a\sec\theta$, $r = a\operatorname{cosec}\theta$, $r = a + b\cos\theta$, $r = a\cos 2\theta$, $r = a\sin 3\theta$, $r = a\theta$
- de Moivre's Theorem, applications to nth roots and multiple angle formula

Unit 7 Logical Reasoning

Section 7.1 Sentences, connectives

Statements, sentences, open sentences, definitions

Logical connectives and quantifiers and their use in proof and disproof

Unit 8 Mathematics of Investment

Unit 9 Statistics and Probability

(An intuitive approach)

Section 5.4 **Groups**

* Definition of group
* Investigation of various transformation groups
* Use of two-dimensional models of regular polygons to establish symmetry group characteristics

Examination of the systems of natural numbers, integers, rational numbers, real numbers and complex numbers to determine which systems form groups

* Examination of sets of functions that form groups ($f \circ g$ defined as $f(g)$ and use of f^{-1} as the group inverse)

Permutation developed as a one-to-one mapping of a set onto itself

Non-commutative groups

* Representation of various transformations by 2×2 matrices, investigation of group properties, transformation groups and the corresponding matrix groups

Identification of the characteristic properties of rings and fields, examples of rings and fields

* Transformations of a regular tetrahedron and cube

Unit 6 Complex Numbers and Polar Coordinates

Portions of this unit can be related to Units 2 and 3: complex numbers with the intersections of curves, the graphs of some equations in polar form with the conics, and de Moivre's Theorem with trigonometry and polar coordinates. Topics marked with an asterisk can be adapted to such a development

Section 6.1 **Complex numbers**

* Complex numbers as ordered pairs and in the form $a + bi$

Field properties of the complex numbers

* Solutions of quadratic equations

Geometric form of complex numbers, absolute value, complex conjugates

* Polar form of complex numbers

Section 6.2 **Polar coordinates**

* Establishment of the correspondence for the point P between the rectangular coordinates (x, y), the polar coordinates (r, θ), and the rectangular components of the vector OP
* Practice at conversions from any one of these pairs to the other two
* Practice in graphing of relations expressed in polar form, particularly $r = a$, $r = a \cos \theta$, $r = a \sin \theta$, $r = a \sec \theta$, $r = a \operatorname{cosec} \theta$, $r = a + b \cos \theta$, $r = a \cos 2\theta$, $r = a \sin 3\theta$, $r = a\theta$
* de Moivre's Theorem, applications to n'th roots and multiple angle formula

Unit 7 Logical Reasoning

Section 7.1 **Sentences, connectives**

Statements, sentences, open sentences, definitions

Logical connectives and quantifiers and their use in proof and disproof

Unit 8 Mathematics of Investment

Unit 9 Statistics and Probability

(An intuitive approach)

Calculus

Honour Graduation Level

Outline of Topics

This outline of topics contains mathematics taken without [illegible] curriculum [illegible] local planning of courses [illegible]

Although no optional topics are specifically listed adaptations may be made to suit the interests and needs of the students The course may be adapted by variation of the rate of the development by extension of the topics or related experiences in other courses in mathematics, science and technology and by introduction of related ideas such as absolute convergence and application of the binomial theorem to series of which is specifically mentioned in the outline In Unit 4 only those aspects of complex numbers and polar coordinates which can be related to other topics in the course need be emphasized the structure of the set of complex numbers is more closely related to *Algebra* Students not taking *Relations and Functions* still need the Students ideas related to parts of the first three units of the course in order to understand and apply some of the concepts of Calculus The level of sophistication and depth of treatment however should maintain a standard traditionally associated with grade 13 mathematics

---

Unit 1 Slopes and Simple Derivatives

This unit begins an intuitive approach to calculus placing emphasis on the graphical and physical interpretation of the derivative which in this treatment is defined as a new function f' such that $f'(x)$ equals the slope of the graph of the function at the point $(x, f(x))$

Section 1.1 Limits

An intuitive discussion of the limit of a sequence

The sum of an infinite geometric series

An intuitive approach to the definition of the limit of a function

Section 1.2 Rate of change

Discussion of rate of change for a linear function

The slope of a straight line in terms of [illegible]

Consideration of the problem of determining the (average) rate of change when other than linear functions are involved

Study of the graph of $y = x^2$ to determine the slopes of secants, e.g. the one from $x = 2$ to $x = 2 + \Delta x$ ($= 4 + \Delta x$), a similar study of the function defined by $y = x^3$

Consideration of the problem of determining the instantaneous rate of change

The concept of tangent as the limiting position of secants, application of this concept to the cases of the secants of $y = x^2$ and $y = x^3$ to determine the rate of change (the slope) at specified points, checking of these values by measuring the slopes of the tangents at these points

Similar treatment at specified points on the graphs of equations such as $y = 2x^2 + 3x$, $y = x^3 - 8x$

Section 1.3 Derivatives

Determination of the slope at the point (x, y) for rational functions e.g. the functions defined by

$y = x^4$, $y = x^5$, $y = \frac{1}{x}$, $y = 8x^2 + 6$

$y = 2x^3 + 5x + 1$

The definition of f' (the derivative of f)

The notations $D_x y$, $\frac{dy}{dx}$

Determination of the derivative of $x \to x^n$ where n is a positive integer or zero

Determination of the derivative with respect to x of a sum $u + v$, a product uv and a quotient $\frac{u}{v}$, where u and v are differentiable functions of x

Determination of the derivative with respect to x of u^n where n is a positive integer and u is a differentiable function of x

Assumption of the derivative of u^r with respect to x where r is a rational number and u is a differentiable function of x

Practice in calculation of derivatives involving sums, products, quotients, and functions defined implicitly

The derivative of the functions $x \to \sin(kx + d)$ and $x \to \cos(kx + d)$

Problems in maximum and minimum values and in displacement, velocity, and acceleration where trigonometric functions are involved

---

Unit 2 Applications of Differentiation

Many of the applications appearing in this unit will have been encountered by the pupils in previous experiences in mathematics and science It is the objective of this unit to give practice with these practical problems

Section 2.1 Tangents

Use of the derivative to find the slope of the tangent at a given point on a curve

Use of the derivative to find the equation of the tangent at a given point on a specific curve (particular cases only of circles, ellipses, parabolas and hyperbolas)

Section 2.2 Velocity, acceleration

Problems determining average velocity during specified time intervals and the instantaneous velocity at specified times when given a relation between distance and time

Calculation of $D_t s$ in various equations from physics involving s and t, e.g., $s = 16t^2$, $s = 200t - 16t^2$, interpretation of these derivatives

Consideration of the meaning of instaneous change of velocity and its calculation as $D_t v$

Problems in acceleration

Section 2.3 Second derivatives, extreme value

Consideration of the graphical significance of the second derivative and the interpretation according as its value is positive, negative, or zero

Practice in curve-tracing using the first and second derivatives

The existence of an extreme value (maximum or minimum) of a quadratic function, determination of the extreme value using the first derivative, identification of the extreme value as either maximum or minimum

Extension to other than quadratics

Problems involving maxima and minima

Applications to rate problems

Unit 3 Equations of the form $D_x y = f(x)$

Section 3.1 Solution of differential equations

Solution by inspection of equations of the form $D_x y = f(x)$

Finding the equation of a family of curves with a given slope

Selection of a particular member of the family which satisfies a prescribed condition

Section 3.2 Motion

Finding the velocity–time relation given the acceleration relation

Finding the distance–time relation given the velocity relation or the acceleration relation

Other problems involving distance, velocity, acceleration, and time

Section 3.3 Area

Intuitive development, using a graph, to establish that the function f is the derivative of the area function A, and hence that A may be determined by solving the equation $A'(x) = f(x)$

Calculation of the area bounded by a given curve and given ordinates

Calculation by subtraction (or addition) of the area between two curves

---

Unit 4 Complex Numbers and Polar Coordinates

The topics marked with an asterisk can be related to certain aspects of calculus

Section 4.1 Complex numbers

- Complex numbers as ordered pairs and in the form $u + bi$

  Field properties of the complex numbers

- Solutions of quadratic equations

  Geometric form of complex numbers, absolute value, complex conjugates

- Polar form of complex numbers

Section 4.2 Polar coordinates

- Establishment of the correspondence for the point P between the rectangular coordinates (x, y), the polar coordinates (r, θ), and the rectangular components of the vector OP

- Practice at conversions from any one of these pairs to the other two

  Practice in graphing of relations expressed in polar form, particularly
  $r = a$, $r = u \cos \theta$, $r = u \sin \theta$, $r = u \sec \theta$, $r = a \operatorname{cosec} \theta$, $r = a + b \cos \theta$, $r = a \cos 2\theta$, $r = u \sin 3\theta$, $r = u\theta$

  de Moivre's Theorem, applications to nth roots and multiple angle formula

Algebra

Honour Graduation Level

Outline of Topics

This outline of topics [illegible] mathematics taken without serious [illegible] curriculum S.12C. It is intended to engage in the local planning of a course based on the main matters in the core units and augmented by some of the topics in the optional units.

This course should take into consideration the students earlier experiences with the topics in the outline. This is particularly so for the topics on sets and vectors, in which the students may have quite extensive backgrounds. The linear algebra aspects of this course could be emphasized particularly for students interested in careers in the fields of mathematics, the sciences, engineering, and technology. For these students a broader range of problems could be investigated and the underlying structures of mathematics should be examined. The level of scholarship and depth of treatment should maintain the standards traditionally associated with grade 13 mathematics.

Some of the optional units in this outline are also based in Relations and Functions of Calculus because certain of their aspects are related to the core of these courses. Attention should be accorded in deciding where and with what emphasis these ideas are to be developed

A

Unit 1 Sets, Subsets, and Permutations

Section 1.1 Sets, subsets

Unions, intersections, and complements of sets

Use of Venn diagrams to illustrate the commutative, associative, and distributive laws of intersection and union

The complement of a union and the intersection of the complements, the complement of an intersection and the union of the complements

Study of particular cases to establish the fundamental counting principles

(a) If a finite set A contains r elements and a finite set B contains s elements, then there are rs different ordered pairs (a, b) where $a \in A$ and $b \in B$ (or, equivalently, the Cartesian product $A \times B$ contains rs elements).

(b) If a finite set A contains r elements, a finite set B contains s elements, and the intersection contains t elements, then the union of A and B contains $r + s - t$ elements.

Precision in determining the number of elements in $A \times B$, $A \cup B$, and $A \cap B$.

Extension to more than two sets

Section 1.2 Permutations

Definition of an n-arrangement (permutation) of n things as an ordered selection or arrangement of all of them

The study of particular cases to establish that the number of n-arrangements of n things is $n(n-1)(n-2)(n-3) \ldots (3)(2)(1)$

The particular case where $r = n$: the factorial notation, $n!$

Number of arrangements of n elements not all different

Section 1.3 Combinations

Definition of an r-subset (combination) of a set of n elements as a selection of r of them without regard to order

Development of the number of subsets of a set of n elements as 2^n

Development of the number of r-subsets of a set of n elements

Use of the notation $\binom{n}{r}$

Unit 2 Mathematical Induction and the Binomial Theorem

Section 2.1 Mathematical induction

The method of mathematical induction

Proof by mathematical induction of the sum of the first n terms of the general arithmetic and geometric series

Use of examples to illustrate the properties of sigma notation for summation, namely

$$\sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} a_i = a \sum_{i=1}^{n} a_i, \quad \sum_{i=1}^{n} a(b_i) = a \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Use of mathematical induction to establish the formulas for the sum of the squares and of the cubes of the first n natural numbers

Study of a few examples proved by mathematical induction and by some other method

Study of at least one example which is true for $n = 1, 2, 3, \ldots, k$ and is false for $n = k + 1$

Section 2.2 Binomial theorem

The proof of the binomial theorem for positive integral exponents by the use of the number of r-subsets

Statement of the general term in the binomial expansion

Reference to the use of Pascal's triangle

The proof of the binomial theorem by mathematical induction

A

Unit 3 Vectors

Section 3.1 Vectors as directed line segments

Vectors as directed line segments in 2-space and in 3-space without a coordinate system

Equality of vectors with attention to the fact that only length and direction are relevant

Addition of vectors by use of parallelograms and triangles, subtraction of vectors, the negative of a vector, the zero vector

Multiplication by a scalar

The algebraic properties of vector addition and multiplication by a scalar illustrated by geometrical constructions

Linear combinations of vectors, the conditions that (a) two vectors be collinear, (b) three vectors be coplanar, and (in) three vectors have their end points on a line (internal and external division of a line segment)

Use of vectors to prove some propositions from deductive geometry, e.g.

(a) If two sides of a quadrilateral are equal and parallel, the other two sides are also equal and parallel.

(b) The diagonals of a parallelogram bisect each other.

(c) The line joining the mid-points of two sides of a triangle is parallel to the third side and equal in measure to one-half of it.

(d) The medians of a triangle intersect at a point at two-thirds of the distance from each vertex.

Resultant and equilibrant of forces acting at a point, resultant of velocities

Section 3.2 Vectors as ordered pairs and ordered triples

The Cartesian coordinate system for 2 and 3 dimensions, identification of points with pairs or triples

Definition of addition, subtraction, and multiplication by a scalar for pairs or triples

Verification that pairs and triples satisfy the algebraic properties of vector addition and of multiplication by a scalar

Geometric interpretation of pairs or triples as vectors

Section 3.3 Linear combinations

Lines and planes through the origin as linear combinations of given vectors, collinear and coplanar vectors

Demonstration that for any three vectors in 2-space, at least one is a linear combination of the other two

Development that, for any two non-collinear vectors in 2-space, any vector is a linear combination of them, and extension of this concept to non-coplanar vectors in 3-space

Section 3.4 Dot product

The standard bases $(1, 0), (0, 1)$ and $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

The distance formula and length of line segments in two and in three dimensions

The length of a vector ta

Dot or inner product $a \cdot b$, relation to the cosine law

The algebraic properties of dot product

Angles and projections, perpendicularity

Unit vectors and normalization

Applications to resolution of forces and work

Unit 4 Equations of Lines and Planes

Section 4.1 Lines

Lines and planes in general position

The vector equation of a line in 2-space, the resulting parametric equations of a line in 2-space

The linear equation of a line in 2-space (developed using the inner product)

The vector equation of a line in 3-space, the resulting parametric equations of a line in 3-space

Direction angles, cosines and numbers

Section 4.2 Planes

The vector equation of a plane in 3-space, the resulting parametric equations of a plane in 3-space

The linear equation of a plane in 3-space (developed using the inner product)

Geometric interpretation of the solution set of two or of three linear equations

A

Unit 5 Systems of Linear Equations

Section 5.1 m equations in n unknowns

Solution of m equations (homogeneous and non-homogeneous) in n unknowns by eliminating variables (specific values of m and n not too large)

Inconsistent systems

Section 5.2 Solution by matrices

Definition of the augmented matrix of a system of m equations in n unknowns

Development of the technique of obtaining a row reduced echelon form of the augmented matrix

The solution in parametric form, or the recognition of inconsistency, as read from a row reduced echelon form of the augmented matrix

For $n \leq 3$, interpretation of these parametric solutions as lines or planes as described in Unit 4

Discussion of the $m = n = 3$ case, consistent or inconsistent, in terms of the intersections of planes, by considering the normal vectors in addition to the foregoing techniques

Unit 6 Matrices and Linear Transformations

Section 6.1 Matrices

Definition of an m by n matrix, $A = (a_{ij})$ (m, n arbitrary, but not too large), row and column vectors, definition of AX where X is n by 1

Linear equations of the form $AX = B$

AX as a function of X, statement and proof of linear properties

Section 6.2 Matrices and transformations

Examples of rotation, reflection, projection, shear, identity, zero

Product AB defined as composition

The rule $AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)$.

Justification that AB generalizes AX and that the dot product is a matrix multiplication

Section 6.3 Properties

Associativity, non-commutativity, identity and zero divisors

Definition of A^{-1} and its calculation, interpretation of A^{-1} as the matrix of the inverse of the linear transformation with matrix A, non-invertible matrices

Unit 7 Complex Numbers and Polar Coordinates

Section 7.1 Complex numbers

Complex numbers as ordered pairs and in the form $a + bi$

Field properties of the complex numbers

Solutions of quadratic equations

Geometric form of complex numbers: absolute value, complex conjugates

Polar form of complex numbers

Section 7.2 Polar coordinates

Establishment of the correspondence for the point P between the rectangular coordinates (x, y), the polar coordinates (r, θ), and the rectangular components of the vector OP

Practice in conversions from any one of these pairs to the other two

Practice in graphing of relations expressed in polar form, particularly, $r = a$, $r = a \cos \theta$, $r = a \sin \theta$, $r = a \sec \theta$, $r = a \operatorname{cosec} \theta$, $r = a + b \cos \theta$, $r = a \cos 2\theta$, $r = a \sin 3\theta$, $r = a\theta$

de Moivre's Theorem, applications to nth roots and multiple angle formula

Unit 8 Examples of Groups

Section 8.1 Groups

Definition of group

Investigation of various transformation groups

Use of two-dimensional models of regular polygons to establish symmetry group characteristics

Examination of the systems of natural numbers, integers, rational numbers, real numbers and complex numbers to determine which subsets are groups

Examination of sets of functions that form groups ($f \circ g$ defined as $f(g)$ and use of f^{-1} as the group inverse)

Permutation developed as a one-to-one mapping of a set onto itself

Non-commutative groups

Representation of various transformations by 2×2 matrices, investigation of group properties, transformation groups and the corresponding matrix groups

Section 8.2 Rings and fields

Identification of the characteristic properties of rings and fields

Examples of rings and fields

Transformations of a regular tetrahedron and cube

Unit 9 Probability

This unit is designed to provide an introduction to the topic of probability. The development of the topic is restricted to situations that yield only a finite number of possible outcomes. Solutions of problems may be by the use of tree diagrams or by the use of the sample space

Section 9.1 Basic principles

The definition of the probability P of an event where the total number of outcomes is finite

Practice in writing the set of all possible outcomes

Practice in calculating the probability of an event

Interpretation of solutions indicated by $P = 0$ and by $P = 1$

Discussion of odds

Binomial distribution

Section 9.2 Mutually exclusive events, and otherwise

Use of a Venn diagram to illustrate two events, A and B, that are not necessarily mutually exclusive

Development of the probability that A or B will occur, that is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Derivation of $P(A \cap B) = 0$ and of $P(A \cup B \cup C) = P(A) + P(B) + P(C)$, where A, B, and C are mutually exclusive events

Unit 10 Logical Reasoning

This unit may be interwoven with Unit 1 in developing the properties related to sets.

Section 10.1 Sentences, connectives

Statements, sentences, open sentences, definitions

Logical connectives and quantifiers and their use in proof and disproof

Mathematics of Investment

Outline of Topics

This outline of topics contains units of mathematics taken without revision from *Curriculum RP-33*. It is intended for use in the local planning of a course, subject to considerable adaptation to the students' earlier experiences. Since some topics listed here appear in earlier or parallel courses, care should be taken to avoid unnecessary repetition. This course may be altered or augmented to include introductory ideas and business applications of functions, systems of equations, linear programming, statistics, probability, vectors, matrices, derivatives, and other mathematical concepts currently used in the business world.

MI

Unit 1 Series

Arithmetic and geometric series; the infinite geometric series; the sum of the squares of the natural numbers; related series.

Unit 2 Logarithms

Review of exponent laws for positive integral exponents; definition and use of powers with fractional, zero, and negative exponents; definition of logarithm to the base 10; use of tables for finding the number having a given logarithm; use of logarithms for calculating products, quotients, powers and roots; problems with computations in which the use of logarithms is advantageous.

Unit 3 Interest

Compound interest equation; present value and true discount; nominal and effective rates of interest; construction of interest tables.

Unit 4 Annuities

Definition; kinds of annuities; development of formulas for amount and present value of an ordinary annuity, an annuity due and a deferred annuity; algebraic relation of the formulas used in the annuity tables; annuities at effective rates.

Unit 5 Payment of Debts

Sinking fund method; periodic sinking fund charge and the total periodic charge for interest and sinking fund; use of schedule; amortization method; principal and interest in equal periodic instalments; use of schedule; application to retirement of serial bonds of definite denomination.

Unit 6 Depreciation

Straight line method; constant percentage of diminishing book value method; sinking fund method; annuity method; capitalized cost applied to perpetuities with payment at intervals of more than one year; equivalent costs of assets of different estimated useful lives.

Unit 7 Bond Valuation

Kind of bonds; bond rate and yield rate; factors affecting yield rate; use of yield rate to determine purchase price of straight term bonds by (a) separate valuation of principal and interest, (b) excess or deficit of income; amortization of premium and accumulation of discount; schedules and book values; bonds purchased between dividend dates; use of Makeham's formula; use of bond tables to determine price and yield rate; serial and annuity bonds.

Unit 8 Stocks

Comparison of bonds and stocks; purchase and sale of shares; stock market reports; brokerage; brokerage accounts; transfer taxes; dividends and yield rates.

Unit 9 Variation

Definition of direct, inverse, and joint variation; treatment limited to the fundamental theorems and exercises thereon; graphical representation of variation.

Unit 10 Functions

Examples of functions defined by expressions, by statements, by graphs, by tables of values; formal definition of function; linear functions; graphs of linear functions; determination of a linear function from two known values.

Unit 11 Permutations and Combinations

Fundamental theorem; number of permutations of (a) n different things taken r at a time, (b) n different things taken r at a time if each may be repeated any number of times, (c) n things taken all at a time if some are alike; number of combinations of n different things taken r at a time.

Unit 12 Binomial Theorem

Proof for positive integral exponents; expansion of binomials; approximations with special application to the function $(1 + r)^n$.

Unit 13 Life Insurance

Definition of probability; application to mortality table; expectation of life; life annuities; table of commutation functions; life insurance—whole life, limited payment life, term, endowment; determination of net single premium; net annual premium; natural premium and reserve; gross premium; protection and investment features of insurance.

Unit 1 — Series

Arithmetic and geometric series; the infinite geometric series; the sum of the squares of the natural numbers; related series

Unit 2 — Logarithms

Review of exponent laws for positive integral exponents; definition and use of powers with fractional, zero, and negative exponents; definition of logarithm to the base 10; use of tables for finding the number having a given logarithm; use of logarithms for calculating products, quotients, powers and roots; problems with computations in which the use of logarithms is advantageous

Unit 3 — Interest

Compound interest equation, present value and true discount; nominal and effective rates of interest; construction of interest tables

Unit 4 — Annuities

Definition; kinds of annuities; development of formulas for amount and present value of an ordinary annuity, an annuity due and a deferred annuity; algebraic relation of the formulas used in the annuity tables; annuities at effective rates

Unit 5 — Payment of Debts

Sinking fund method; periodic sinking fund charge and the total periodic charge for interest and sinking fund, use of schedule, amortization method, principal and interest in equal periodic instalments, use of schedule; application to retirement of serial bonds of definite denomination

Unit 6 — Depreciation

Straight line method; constant percentage of diminishing book value method; sinking fund method; annuity method; capitalized cost applied to perpetuities with payment at intervals of more than one year; equivalent costs of assets of different estimated useful lives

Unit 7 — Bond Valuation

Kind of bonds; bond rate and yield rate; factors affecting yield rate; use of yield rate to determine purchase price of straight term bonds by (a) separate valuation of principal and interest, (b) excess or deficit of income; amortization of premium and accumulation of discount; schedules and book values; bonds purchased between dividend dates; use of Makeham's formula; use of bond tables to determine price and yield rate; serial and annuity bonds

Unit 8 — Stocks

Comparison of bonds and stocks; purchase and sale of shares; stock market reports; brokerage, brokerage accounts; transfer taxes; dividends and yield rates

Unit 9 — Variation

Definition of direct, inverse, and joint variation; treatment limited to the fundamental theorems and exercises thereon; graphical representation of variation

Unit 10 — Functions

Examples of functions defined by expressions, by statements, by graphs, by tables of values; formal definition of function; linear functions; graphs of linear functions; determination of a linear function from two known values

Unit 11 — Permutations and Combinations

Fundamental theorem, number of permutations of (a) n different things taken r at a time, (b) n different things taken r at a time if each may be repeated any number of times, (c) n things taken all at a time if some are alike; number of combinations of n different things taken r at a time

Unit 12 — Binomial Theorem

Proof for positive integral exponents; expansion of binomials; approximations with special application to the function $(1 + i)^n$

Unit 13 — Life Insurance

Definition of probability; application to mortality table; expectation of life; life annuities; table of commutation functions; life insurance–whole life, limited payment life, term, endowment; determination of net single premium, net annual premium; natural premium and reserve; gross premium; protection and investment features of insurance

Developing Courses

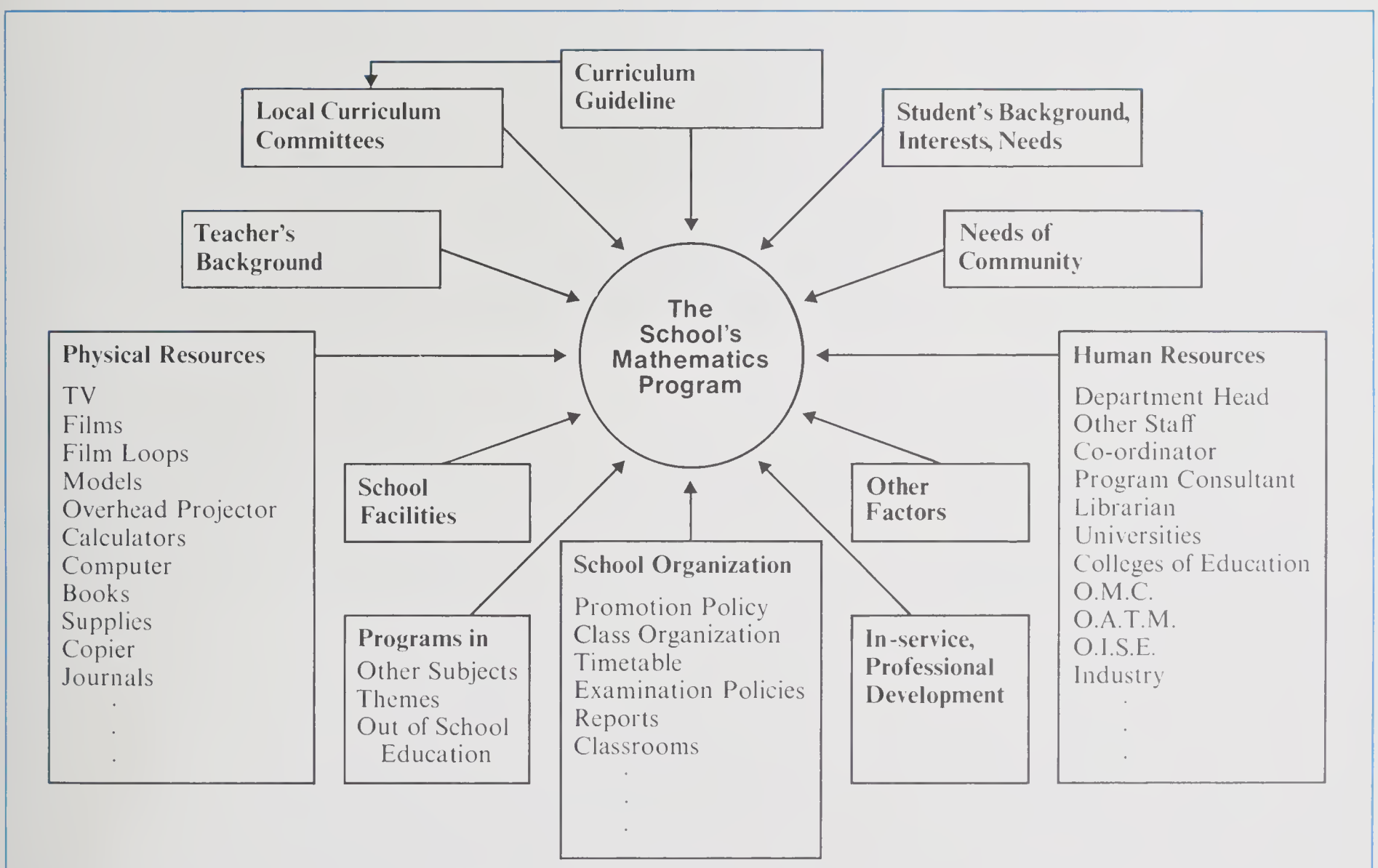
Introduction

The belief that education should be regarded as a process continuing throughout one's lifetime is widely advocated today. The world is changing so rapidly that many of the skills of yesterday are now obsolete, and the pace of these changes is increasing. Thus, today's student needs an education that will prepare him to adapt to his changing environment by a process of continuous self-education. The need is greater now than ever before for an understanding of basic principles and their applications. The emphasis should be on education, not training.

These conditions challenge the school to develop a program that will encourage responsibility through maximum involvement of the student in the curriculum. To achieve this, the program should be current and relevant to the student. It should provide opportunities for progress at a rate largely determined by his own interest, effort, and capabilities. His education, to a considerable degree, should be a process of free enquiry which encourages him to think, to search intelligently for understanding and truth, and to pursue with zeal his individual interests.

Wisdom that grows with experience is essential if a balance is to be maintained between the proliferation of educational forces at play in the community and the foundation of significant ideas needed for a rational curriculum. The student's individual curricular pursuits should blend with a basic program to provide him with the background needed for future studies in a wide variety of areas.

The development of effective courses requires a careful consideration of the general aims of education today, weighed with the objectives of the course and the many human and physical factors that affect its implementation. Some of the considerations affecting aims have been discussed earlier in this document. Specific objectives can be developed locally from the outlines of topics and other related materials. Many of the factors that should be considered during the planning stages are illustrated below.



A Diversified Program

Earlier in this document a distinction was made between the emphases to be developed in courses in *Applications of Mathematics* and courses in the *Foundations of Mathematics*. This distinction is only the first step towards establishing a diversified mathematics program. Within each of these programs a number of courses based on each outline of topics is possible. Such courses should be consistent with the background knowledge of the students, with their academic interests, with their career plans, and with the expectations of the community.

There will be occasions where all students in a class have similar goals. In these cases the course is easily geared to their requirements. For example, the class may be composed of students who are all interested in business mathematics, or in preparing for the study of mathematics at the university level, or in technology at the colleges of applied arts and technology. For these classes, individualizing the curriculum implies that the students will have the opportunity to study their interests in depth and to advance as rapidly as they desire.

On the other hand, it is more likely that classes will be heterogeneous in their interests. In these instances it is desirable that a variety of interpretations of the course be implemented, by varying its scope as described in the next section. It is probable that some students will need remedial work as a basis for understanding the core topics, whereas others, having a good grasp of the core, will be ready for in-depth investigations related to it. While the former group are consolidating the fundamental ideas, the latter group can be profitably engaged with other approaches to the same topics, special projects, group investigations, seminars, essays, problem-solving, or individual study. Obviously, such a multi-faceted program cannot be implemented through teaching based on uniform presentations and standard expectations for all students.

Scope

The scope of a course can be varied in the selection of topics for investigation, the order of investigation, the breadth and depth of study of the topics, the approaches, and the methodologies. These variables are discussed below.

Selection of Topics

The core program is a suggested outline of topics with which all students should be familiar by the time they complete the course. Because the topics in it are basic to the development of mathematics as a whole, it is not likely that it needs to be modified. The classroom development of the core, however, can be varied to suit the students.

The optional and supplementary topics provide ideas that suggest how a course may be adapted to fit local conditions. They, too, can be developed in a variety of ways.

Order, Timing

The organization of each outline into units, sections, and topics is not intended to indicate the optimal sequence of topics. There are many advantages to be gained by integrating the units to emphasize the relationships of various sections and topics.

No indication of suggested timing is given in the outlines because this would tend to prescribe the level of treatment and would assume a uniform 'coverage' of the topics, as well as uniform backgrounds, interests, and needs of students. The time spent on each topic should depend on the objectives of the program and the degree of understanding achieved during its development with the students.

Breadth and Depth of Study

The breadth of study involves not only the number of topics studied, but also the number and variety of situations investigated for each topic. For example, the study of congruence might deal only with a few simple cases of congruent triangles or, at the other extreme, it could deal with investigations of a variety of rectilinear and non-rectilinear two- and three-dimensional figures, using the notions of isometries.

The depth of study depends on the difficulty of the problems investigated under the breadth of topics above, on the precision expected in oral and written work, and on the emphasis placed upon rigour and structure. The boundary between depth of study and breadth of study cannot always be clearly defined.

Approaches

The approach to any topic can vary from a highly intuitive development to a formal one involving rigour, structure, and emphasis on proof.

A further interpretation involves the mathematical vehicle or combination of vehicles by which the topic is developed. For example, geometry can be developed inductively, by a Euclidean approach, by transformations, by vectors, and by combinations of more than one of these approaches, in each instance using synthetic and/or analytic techniques.

Applications

Methodologies

There are many methodologies that can be effective in implementing certain aspects of the program. Ideally, considering the need for students to develop independent study habits, methods that encourage student involvement are the most productive over an extended period of time. Variety in methodology, however, is desirable: it produces a change of pace and will help the student to adapt to different learning situations in later life. The following are some of the methods by which topics can be developed:

individual student assignments, group investigations and discussions, Socratic lessons, lessons using audio-visual aids (such as the overhead projector, ETV, films, and film loops), construction and use of physical models, student-led seminars, lectures by out-of-school resource people, field trips, problem-solving classes, use of calculators and desk-top computers, team teaching, workbooks, programmed learning units, essays and other projects, laboratory situations, small group work sessions using student leaders, remedial sessions, and pre-testing for readiness.

It is desirable that all students be able to relate their experiences in mathematics to other subjects. Mathematics teachers and teachers of other subjects should co-operate to establish needs and specific applications. In this way the applications will be relevant to the students and their special subject interests. The student should experience the process of building mathematical models to represent certain aspects of real-life situations, then work with these models by testing his findings in real situations. Greater encouragement should be given to developing facility in creative problem-solving, ideally related to areas of special interest to the student.

If students are investigating a theme, such as space or pollution, mathematics teachers could offer their services to their colleagues so that programs can be designed to complement one another. Topics may be introduced in the mathematics program in support of the theme project, or the teacher of mathematics may prefer to join forces with the staff of the project to assist the students in understanding its mathematical aspects.

Mathematics teachers should also try to relate their subjects to the total environment. The students should be encouraged to contribute applications of mathematics gleaned from television, radio, films, magazines, newspapers, the world of finance, and personal observation of their environment. Field trips to industrial plants, businesses, the stock exchange, the Ontario Science Centre, museums, planetariums, as well as personal travel, can be used to enrich the students' appreciation of the contributions of mathematics.

In addition, historical anecdotes often help students to become aware of the contributions of mathematics to the advancement of civilization. Mathematical games and puzzles may be used to lighten the classroom atmosphere and at the same time consolidate fundamental skills.

Preparing for Innovations

Generally, in planning for a mathematics program as described above, it would be wise to move steadily in the direction of flexibility, rather than to attempt total change in one major upheaval. Many considerations must be taken into account to achieve this educational ideal: if these are not weighed and planned in advance, frustrations and disappointment may result. A commitment to individual student progress implies a modification of traditional attitudes towards assessment, promotion policies, timetabling, class organization, and the use of the classroom.

Innovation necessitates in-service programs and professional development in order to establish an understanding of aims and objectives and in order to develop suitable techniques for implementing these programs.

Local committees can serve a useful purpose in bringing teachers together to discuss the curriculum in relation to local conditions. Although it may wish to adapt the Department's program suggestions in a general way to suit the county or district level, the function of such a committee should not be to establish uniform courses for all schools in the jurisdiction of a school board. One of its most important roles should be to inventory and even provide resources to assist in the development of the program in the schools. Through these committees, which should focus on curriculum in its broadest sense, plans can be made for in-service programs in both content and pedagogy. The mathematics program consultants of the Department of Education are available to work with these committees upon invitation and to provide resources from a broader base.

Within the school, teachers may do much to assist one another in their professional growth by planning together, sharing developmental ideas, and preparing materials for special projects. Through a personal awareness of the community, inter-class visits, team planning, and department and staff meetings, they can develop insights for improving the program. Communication with other schools is desirable and may be achieved through co-ordinating committees, professional development days, board-sponsored programs and courses, and inter-school visits. Opportunities for professional growth and updating are available through university courses, through summer courses offered by the universities, Ontario Secondary School Teachers' Federation, the colleges of education, and through attendance at conferences sponsored by the Ontario Association of Teachers of Mathematics and other professional bodies. There are many publications that offer relevant information for improving the program. These include a host of current books and articles, the *Ontario Mathematics Gazette*, the *Ontario Secondary School Mathematics Bulletin*, the *Mathematics Teacher*, the *Arithmetic Teacher*, a number of excellent magazines from England, and special publications prepared by the Ontario Institute for Studies in Education and the Ontario Mathematics Commission. Human resources are available from the Department of Education mathematics program consultants in the regional offices and through the co-ordinators and other special staff employed by the boards, universities, colleges of applied arts and technology, colleges of education, and representatives from industry and business.

Evaluation

The introduction of various innovations in the school program (such as non-grading, emphasis on individual student progress and flexible programs in which students pursue individual interests) necessitates a re-evaluation of the means used for assessing student progress. The traditional year-end examination as the primary means of promotion should be supplemented or replaced by more continuous and personal evaluation of the student's progress. Tests based on the topics being investigated can help to build a profile of individual achievement and indicate the degree of readiness to proceed. Student projects, seminars, essays, assignments, and, frequently, day-to-day work, are useful as a basis for evaluation of the student's understanding. Discussions with individual students concerning their work are also valuable. The teacher should be encouraged to exercise professional judgement in the evaluation of the student's knowledge and understanding of the subject, his responsibility and attitudes towards his work, his classroom participation, and his ability to perform on a routine basis without undue pressure.

However, there is still room for the periodic formal examination, especially if it is used to diagnose the individual needs of students. It should test primarily an understanding of common fundamental ideas; at the same time it should provide opportunities for the student to display an understanding of his optional studies.

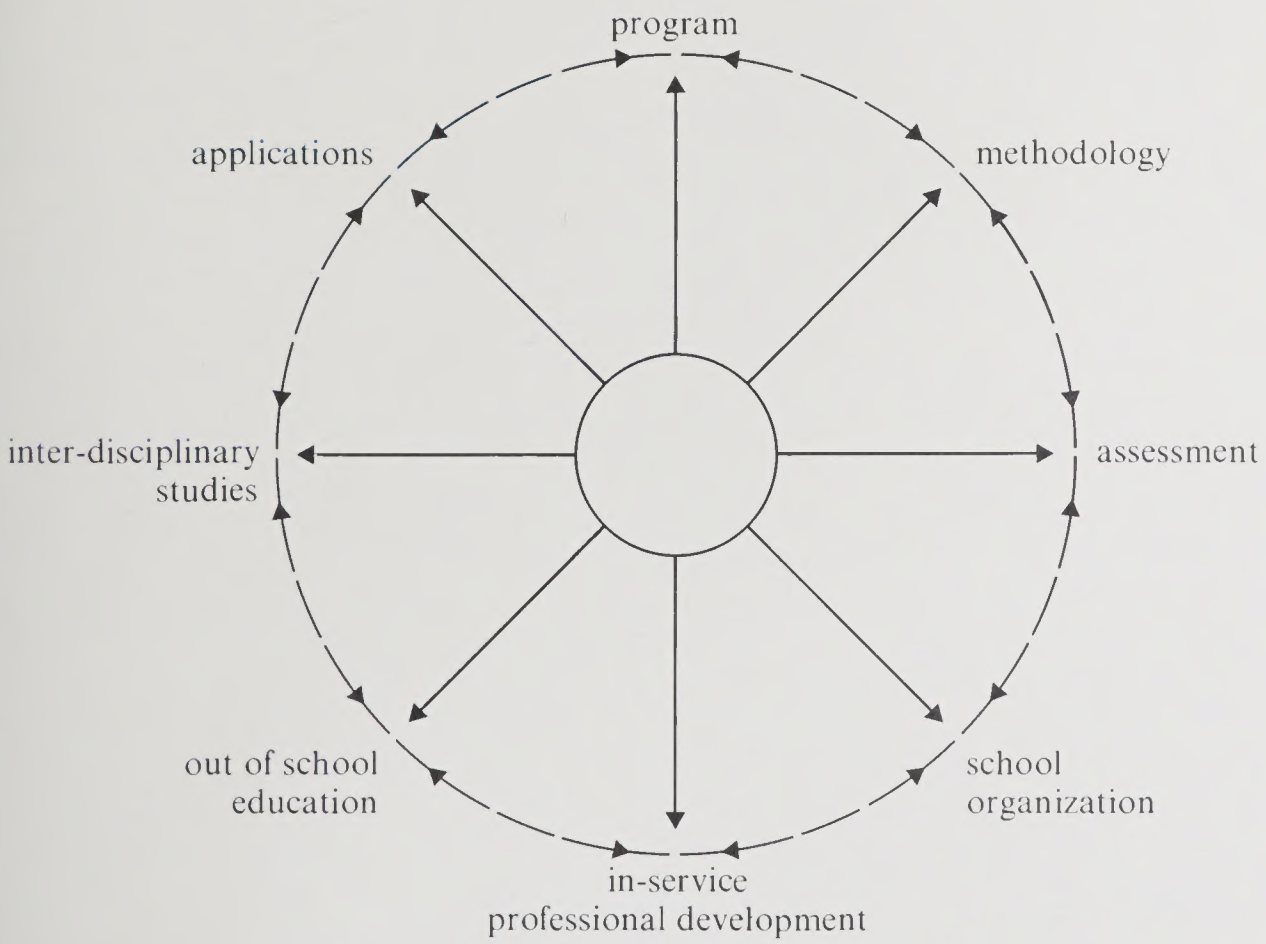
Recognition of achievement in optional areas of the program should be given in the personal records of each student. This has motivational value for the student, diagnostic value for student and teacher, and is significant to the student's future teachers, other educational institutions, and potential employers.

Conclusion

Planning the program in any subject area is a major professional responsibility and one that requires constant reappraisal and modification.

Innovations in the program require careful forethought. In the diagram on this page, the hub represents the state of the curriculum in any school at the present time. It is surrounded by areas in which innovations may occur. The weighting of innovations too strongly in any one direction may produce an imbalance. To restore the balance, a shift in emphasis is needed. In any area of innovation, aspects of all the other areas are interwoven and all must be considered together for the development of a truly vital curriculum.

Innovations: in what direction?



Identification No. 72-1032